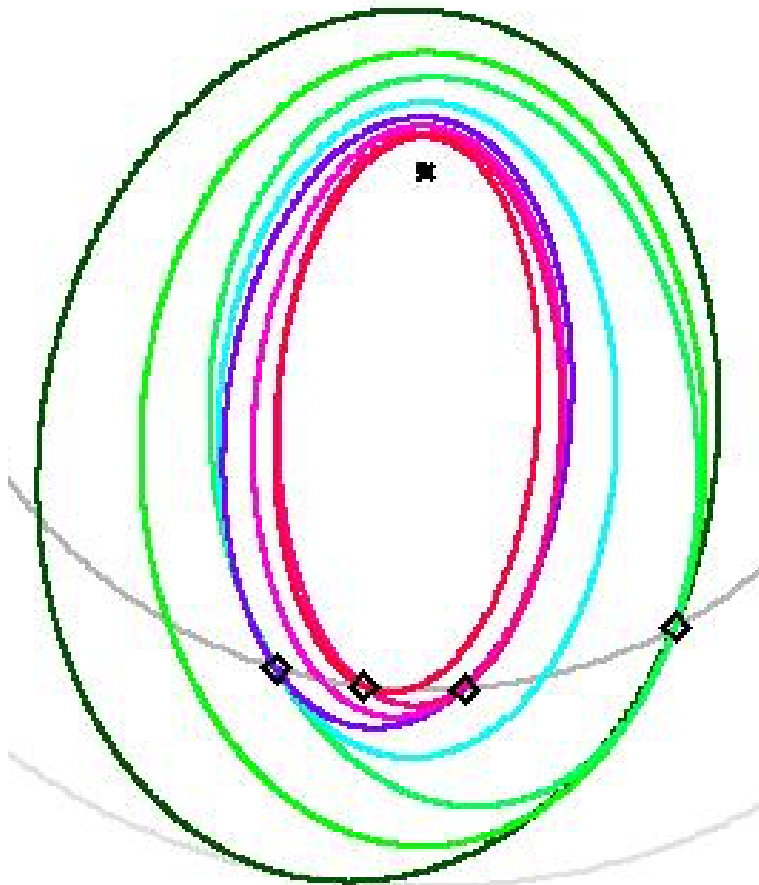
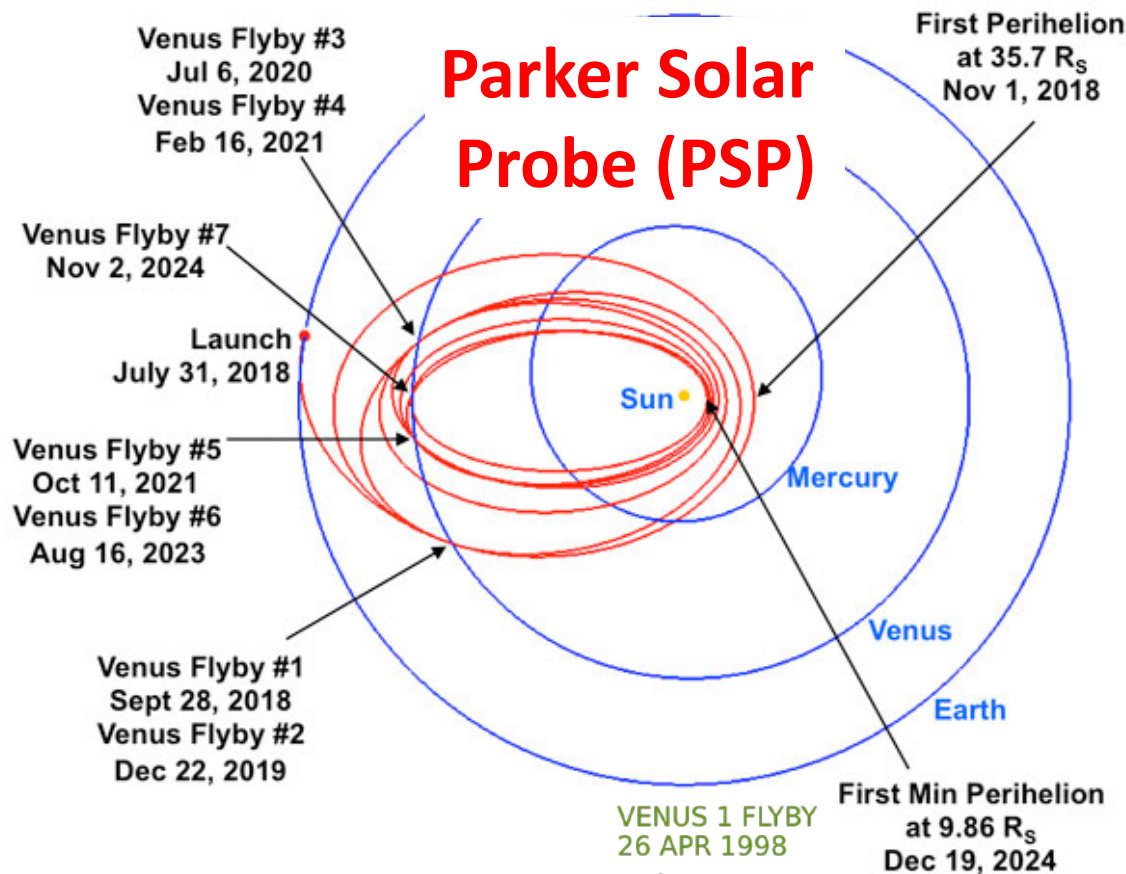


How to get anywhere in the Solar System on a single tank of fuel

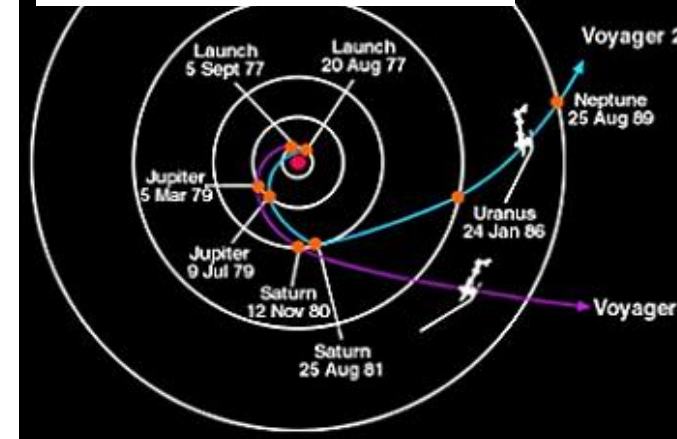


*A dilettante dabbles in
orbital mechanics*

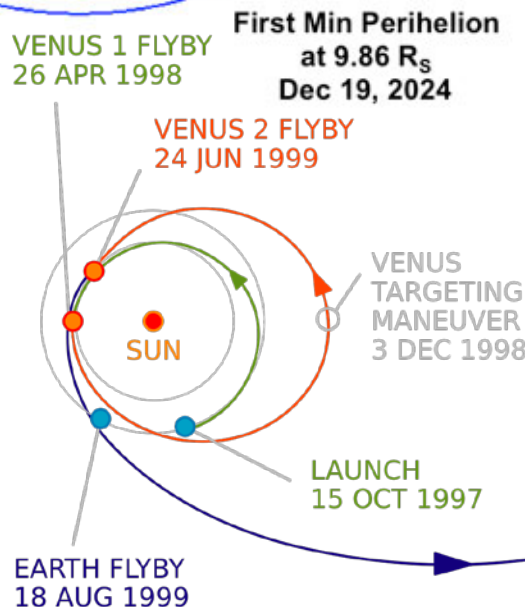
Dana Longcope
Montana State University



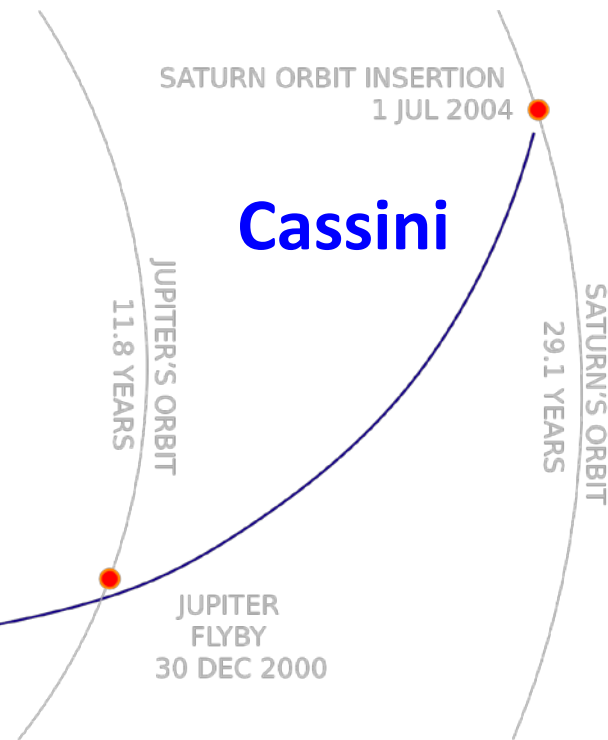
Voyager 1 & 2



Gravity Assists:
 Why?
 How?
 Why so many?



Cassini



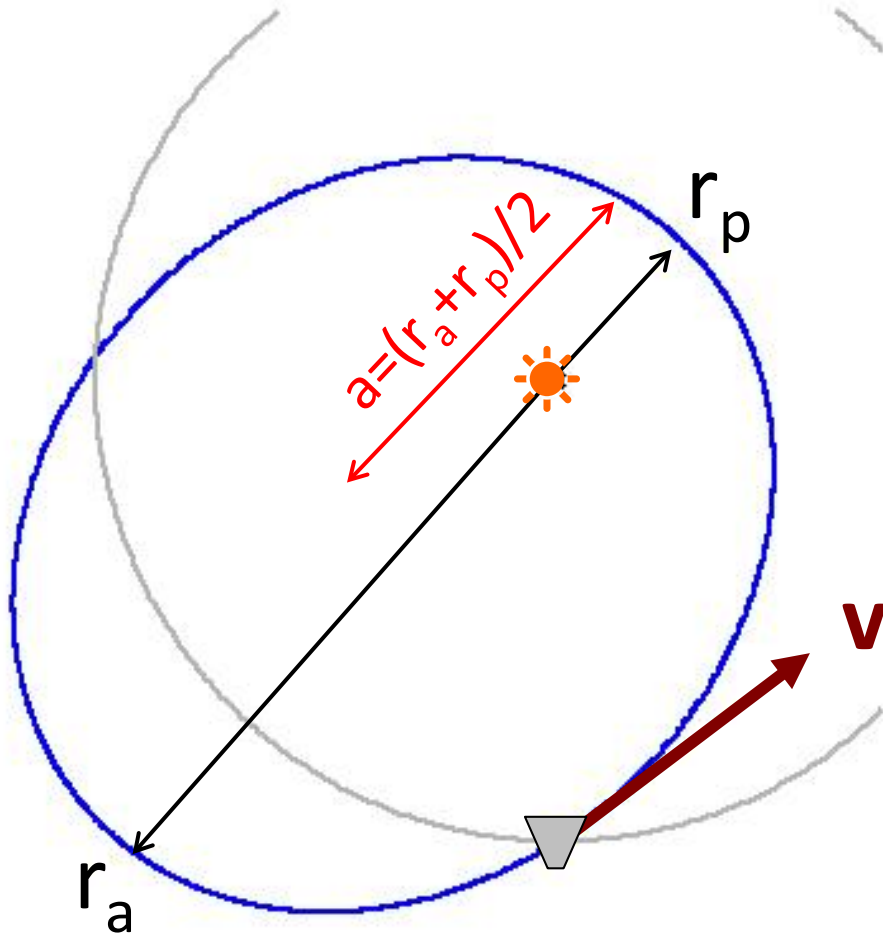
From Fran Bagenal's lecture yesterday:

*Sample Trajectory to Uranus:
Earth-Venus-Earth-Earth-Jupiter
Gravity Assist*

... say what!?

eccentricity: $e = \frac{r_a - r_p}{r_a + r_p}$

$$r_a = (1 + e)a \quad r_p = (1 - e)a$$



S/C orbits Sun
ballistically:

ellipse w/
Sun @ focus

(Kepler's 1st)

eccentricity: $e = \frac{r_a - r_p}{r_a + r_p}$

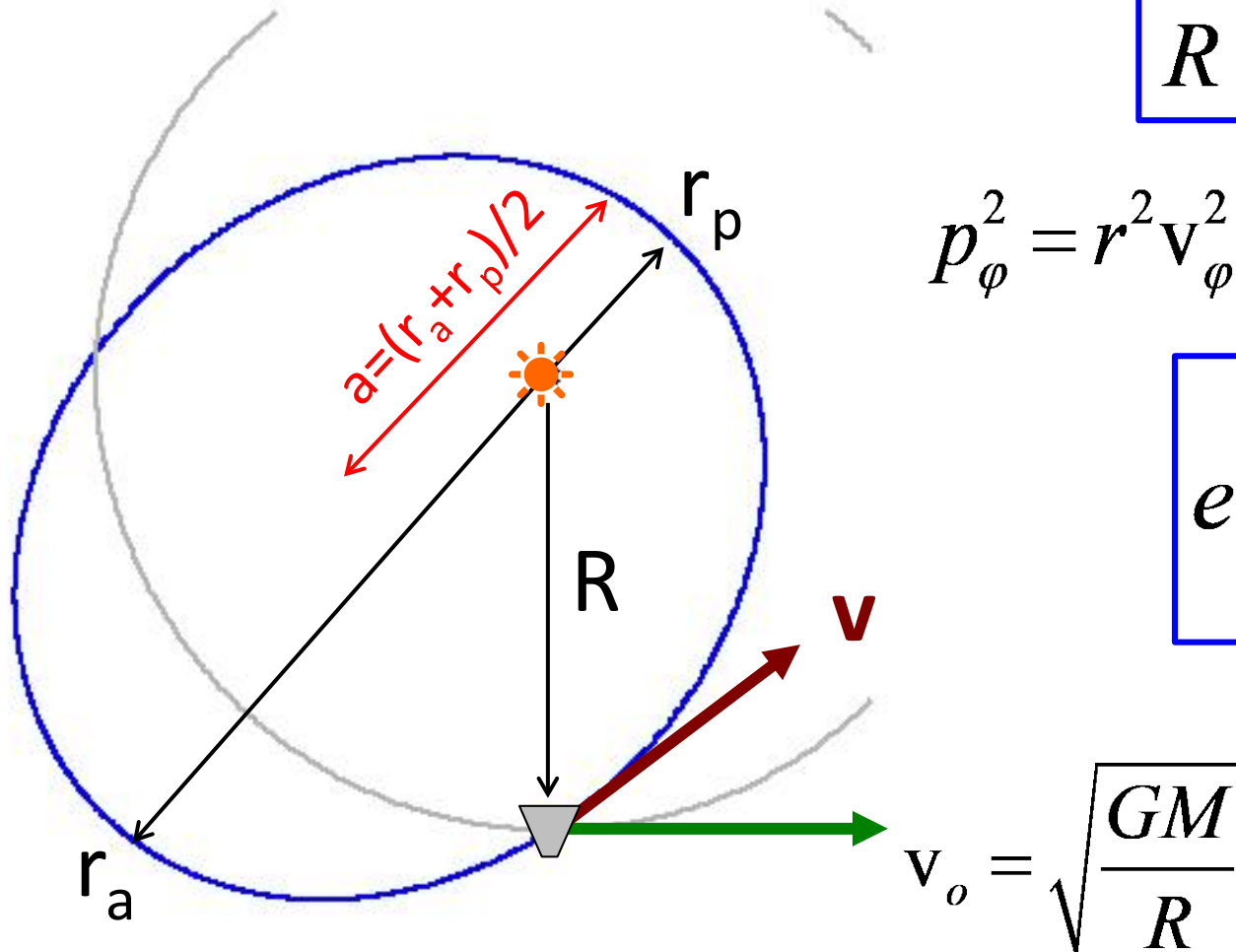
$$E = \frac{1}{2} v^2 - \frac{GM}{r} = -\frac{GM}{2a}$$

$$r_a = (1+e)a \quad r_p = (1-e)a$$

$$\frac{a}{R} = \frac{1}{2 - v^2 / v_o^2}$$

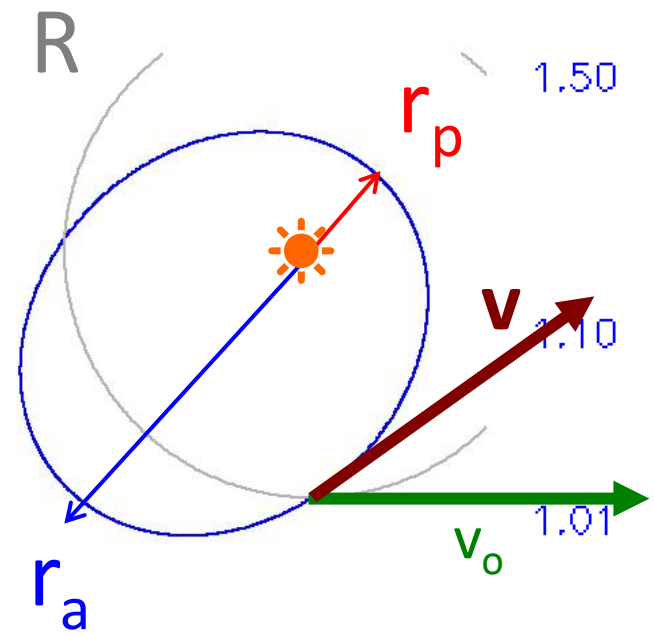
$$p_\phi^2 = r^2 v_\phi^2 = GMa(1-e^2)$$

$$e^2 = \frac{R}{a} \frac{v_\phi^2}{v_o^2} - 1$$



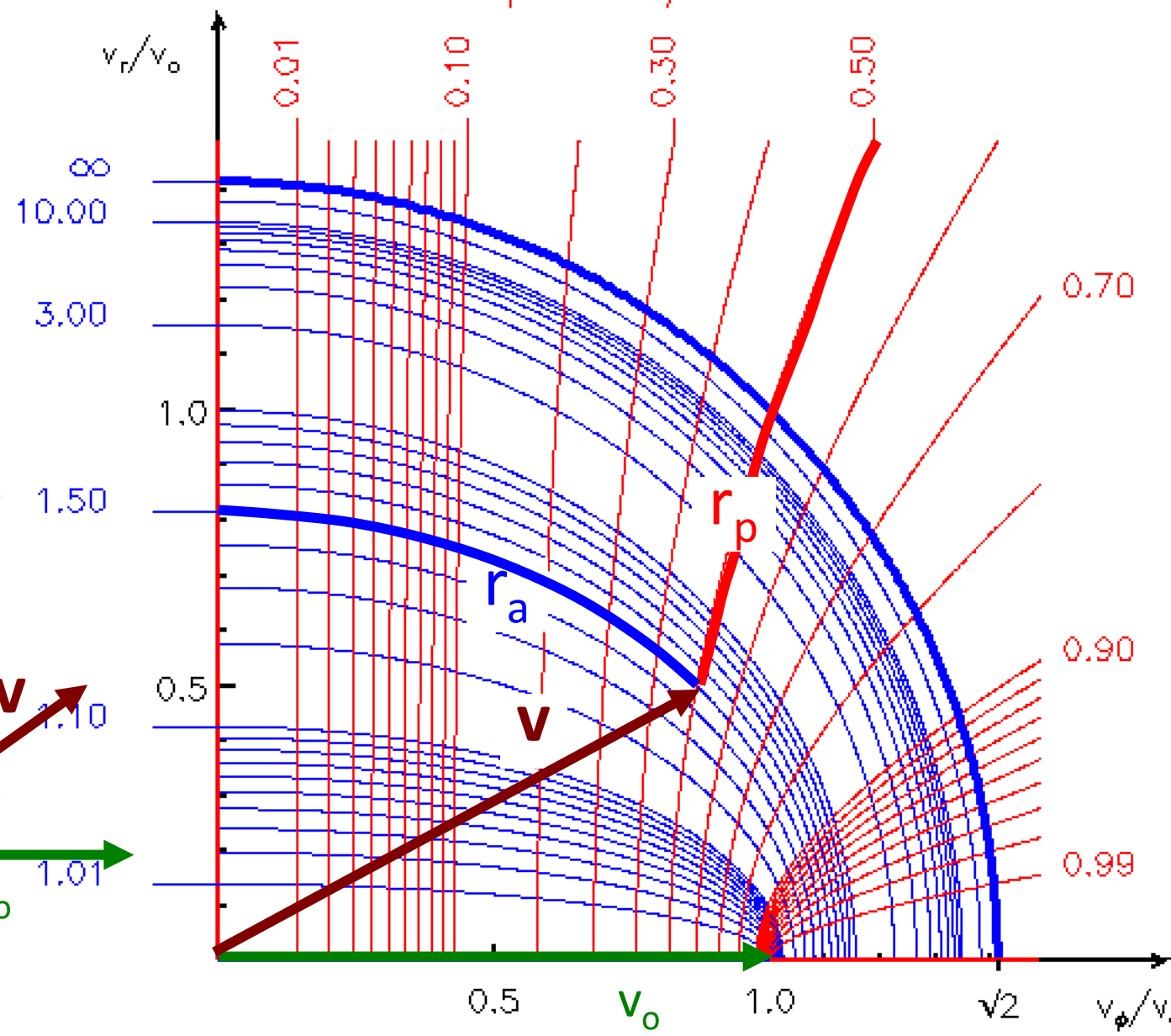
$$\frac{a}{R} = \frac{1}{2 - v^2 / v_o^2}$$

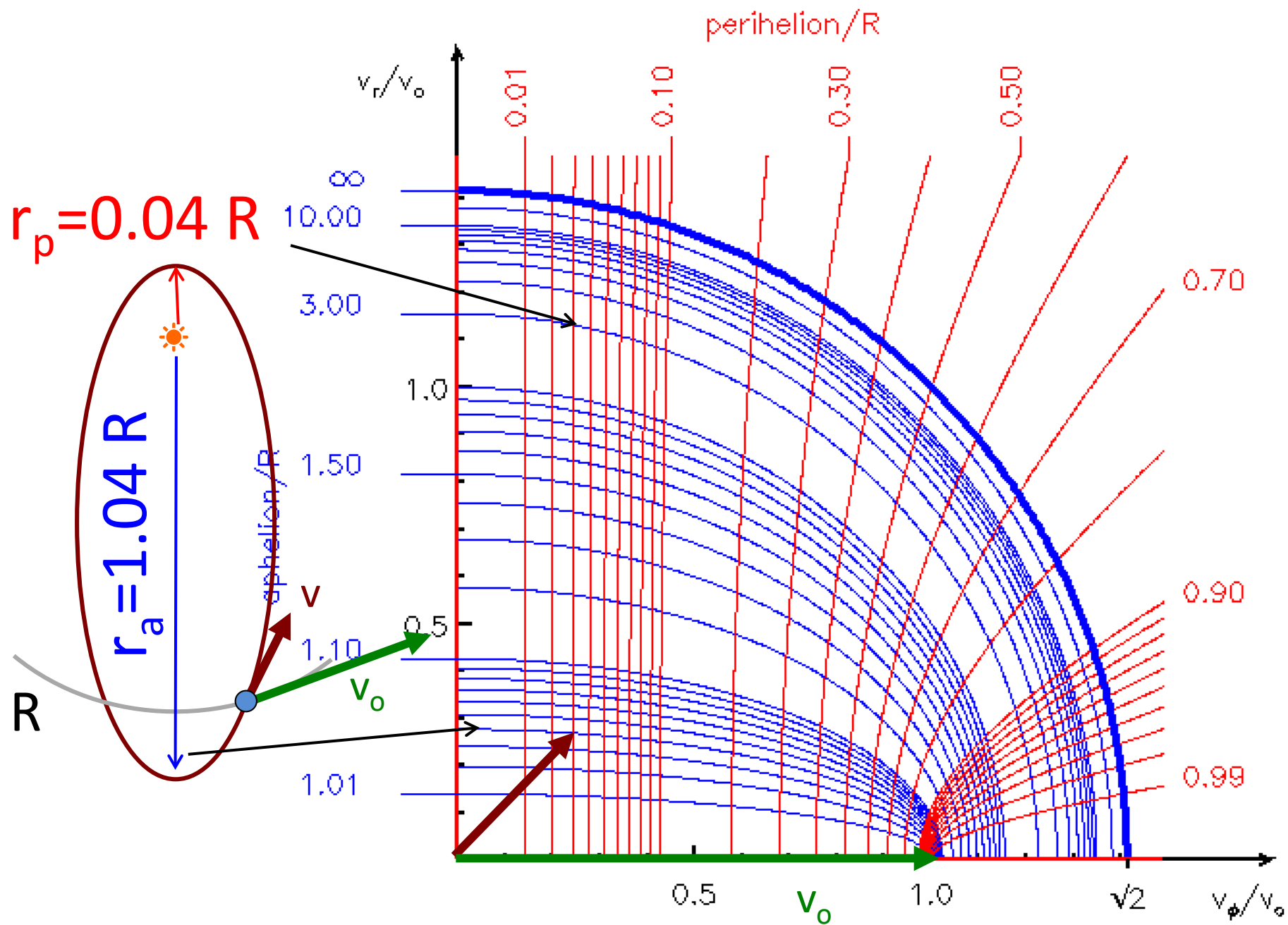
$$e^2 = \frac{R}{a} \frac{v_\phi^2}{v_o^2} - 1$$

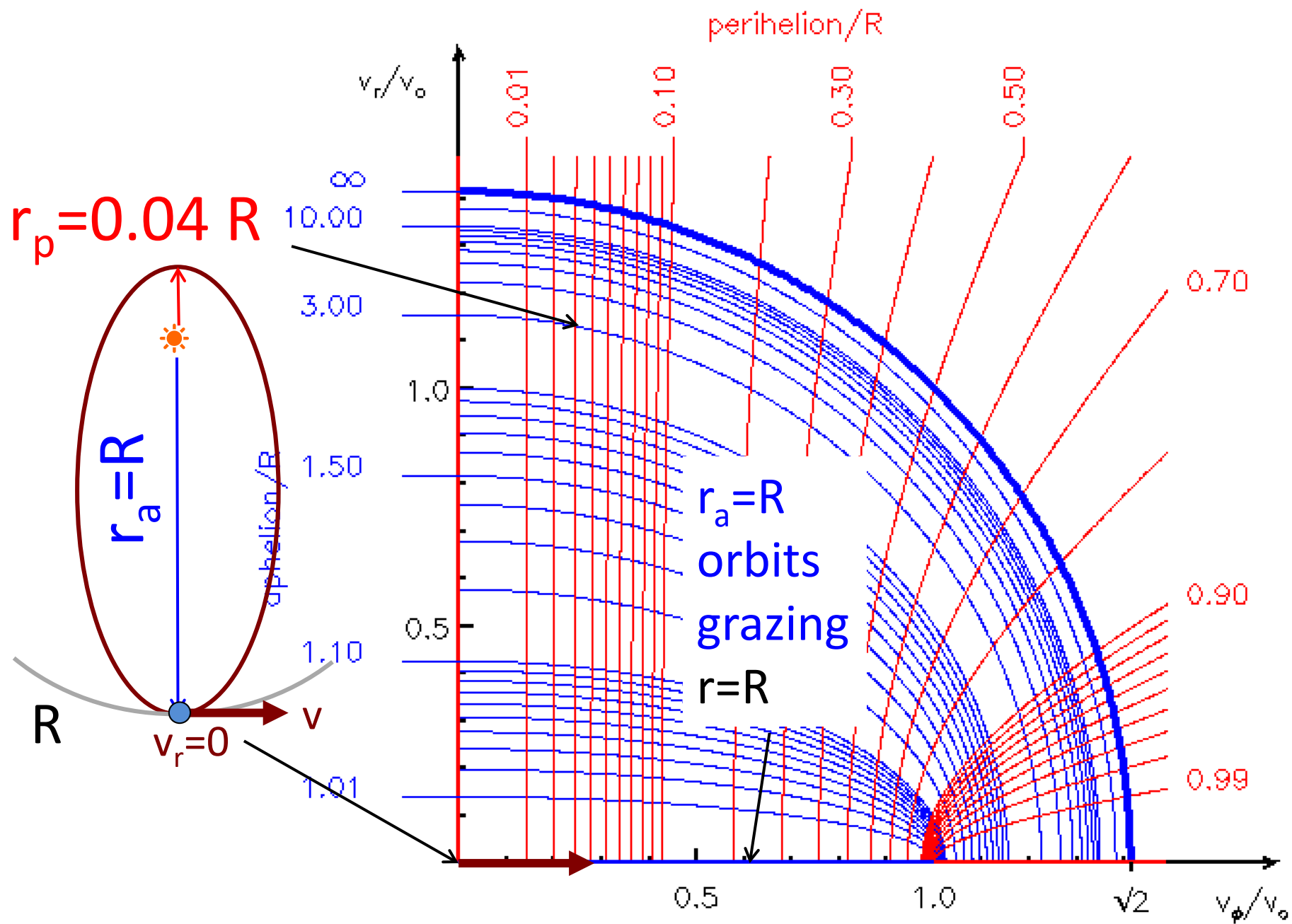


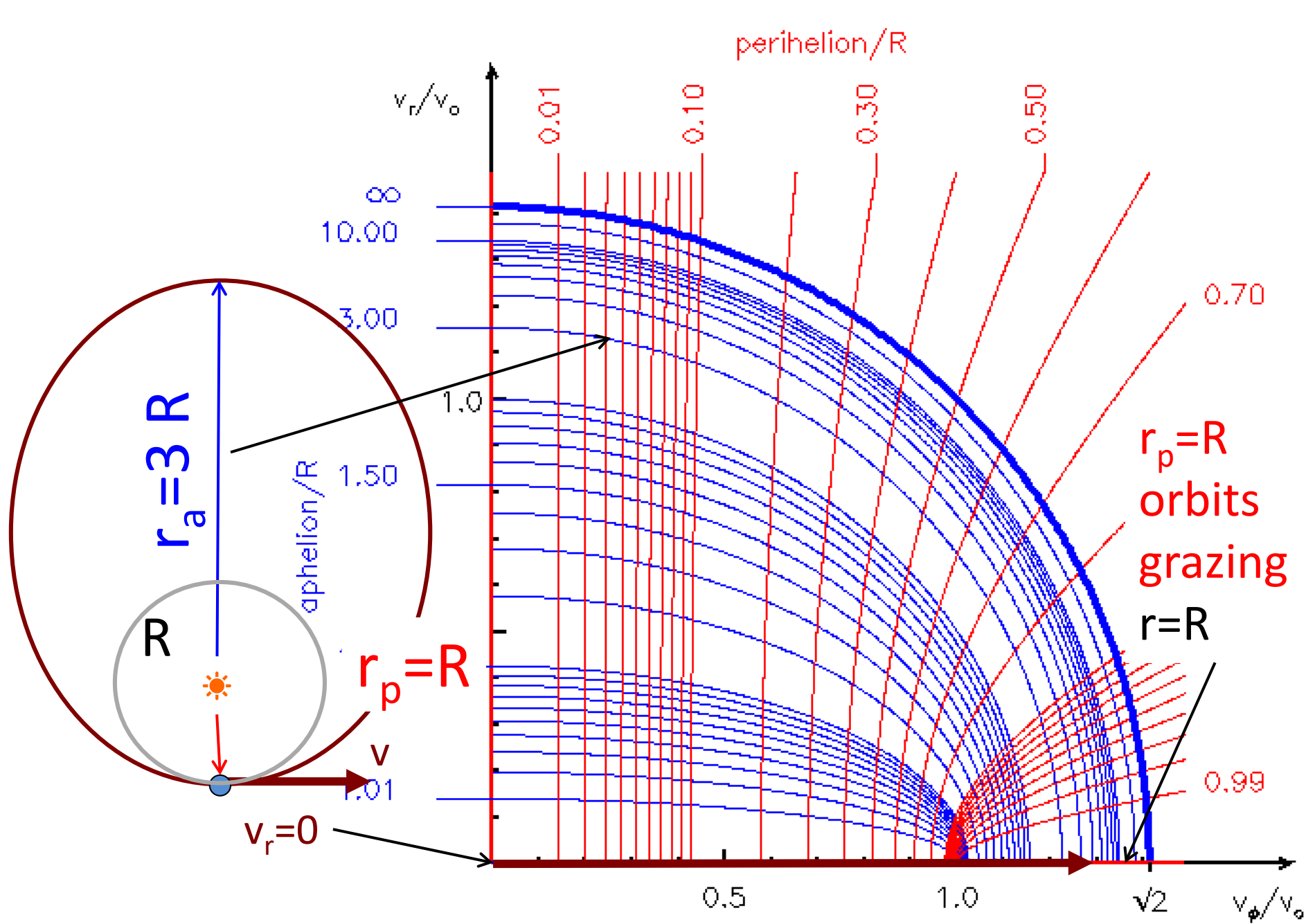
$$r_a = (1 + e)a \quad r_p = (1 - e)a$$

perihelion/R









$$\frac{r_a}{R} = \frac{1+e}{2 - v^2/v_o^2}$$

∞
10.00

aphelion/R
escape orbit

1.10

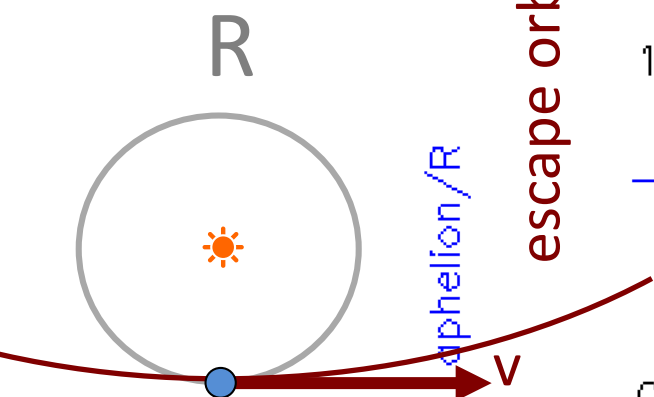
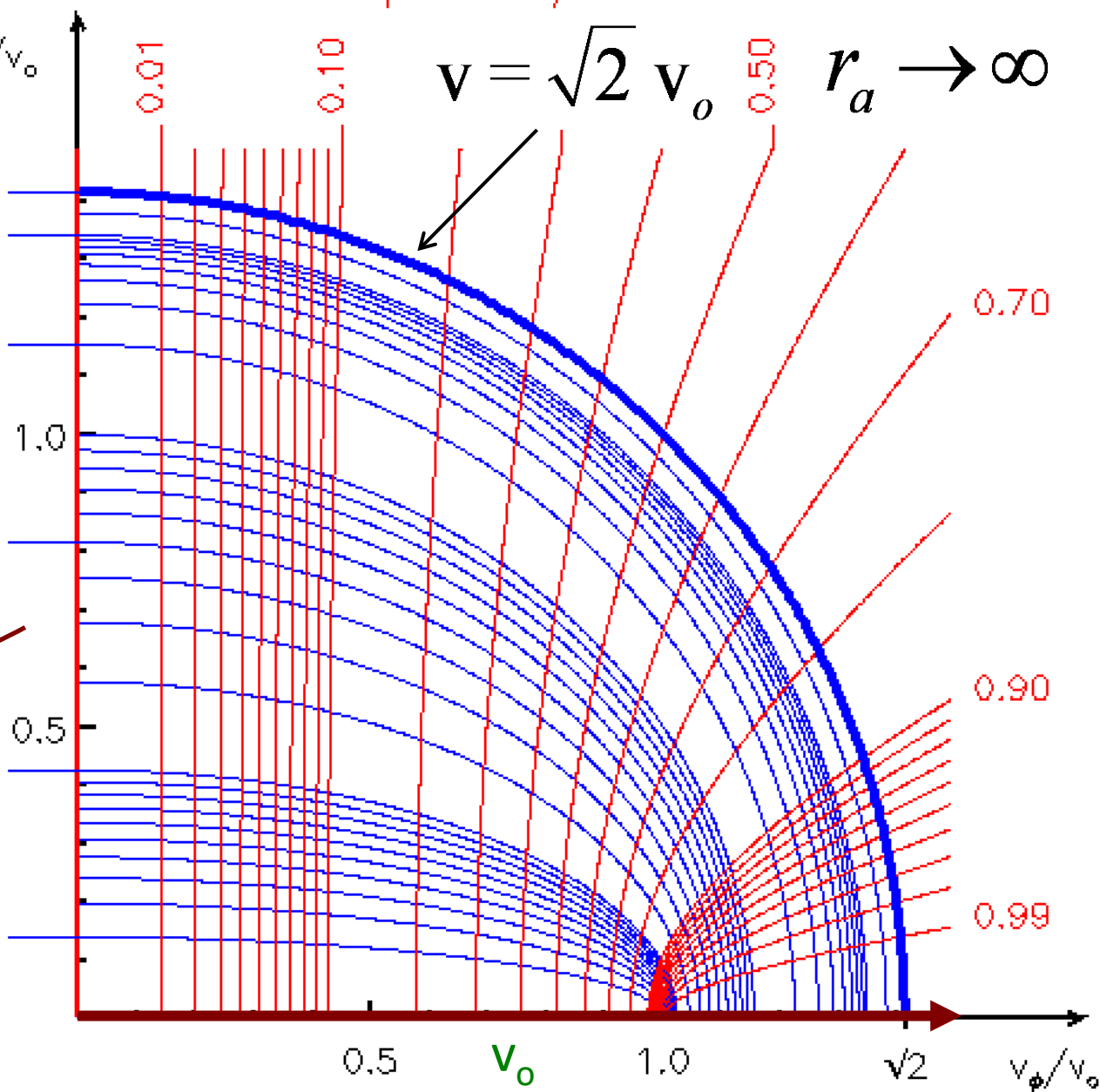
1.01

v_r/v_o

perihelion/R

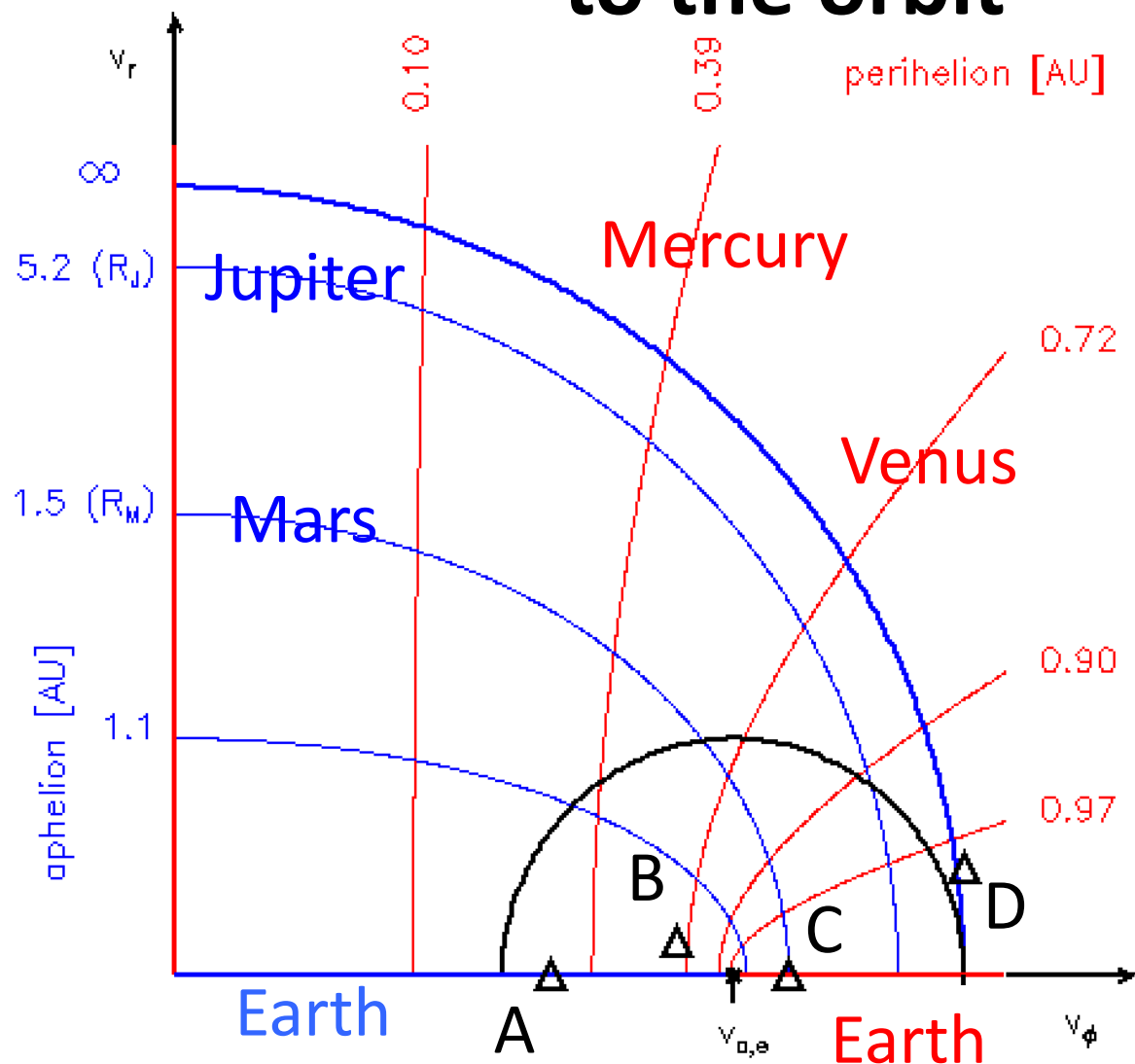
$$v = \sqrt{2} v_o$$

$r_a \rightarrow \infty$



$R=1$ AU

Match the mission to the orbit

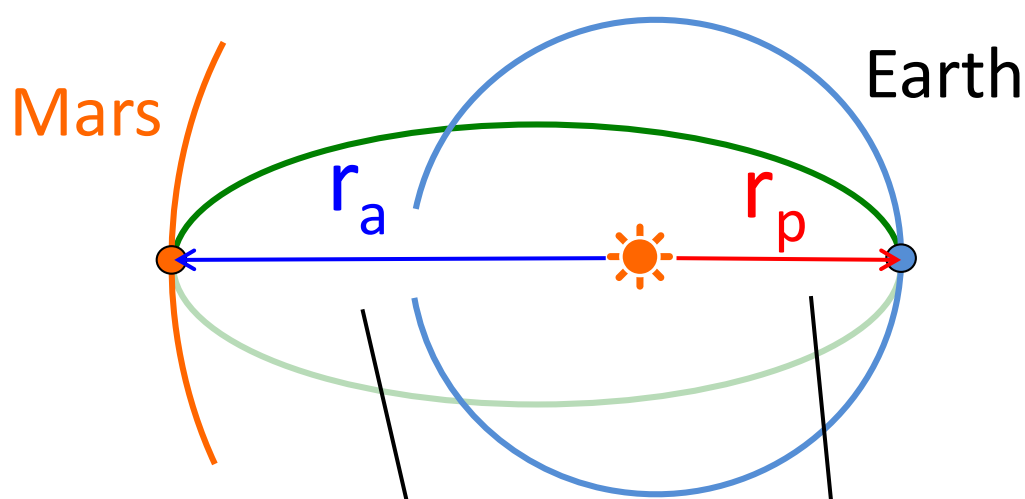


☐ MAVEN:
Earth $\rightarrow$ Mars direct

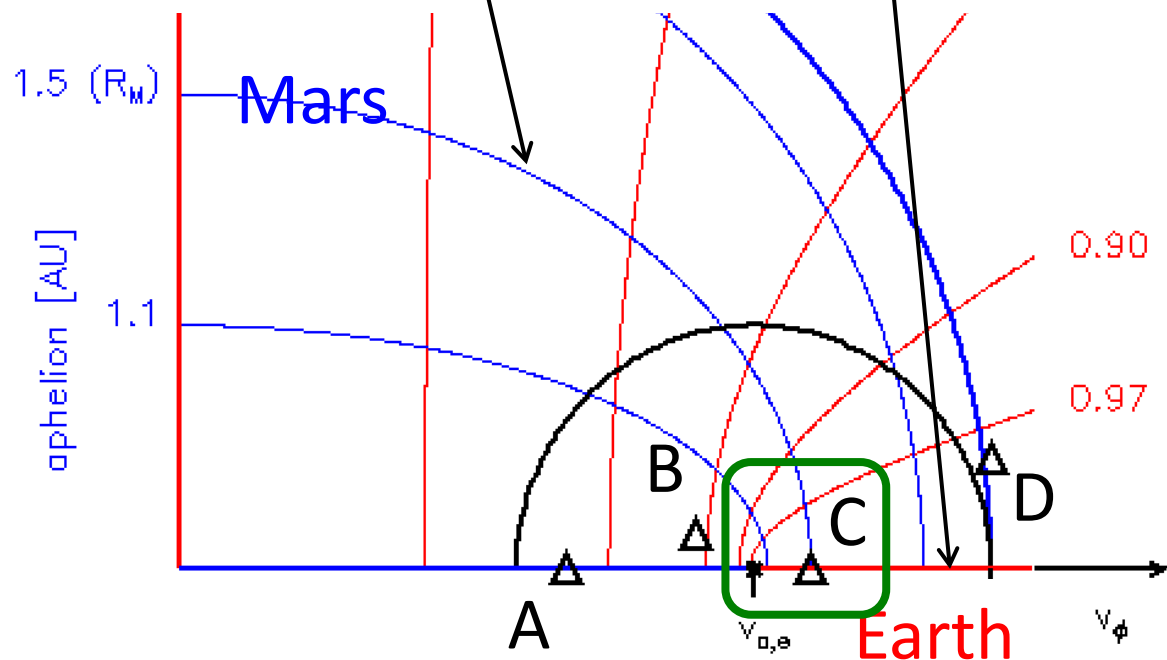
☐ Helios 2:
1976: sampled solar
wind $0.29 < r < 1$ AU

☐ New Horizons:
Earth $\rightarrow$ Jupiter $\rightarrow$ out
passing Pluto
& Ultima Thule

☐ Cassini:
Earth $\rightarrow$ Venus $\rightarrow$ Earth
 $\rightarrow$ Venus $\rightarrow$ Saturn



Hohmann Transfer Orbit



☒ MAVEN:

Earth $\rightarrow$ Mars direct

☐ Helios 2:

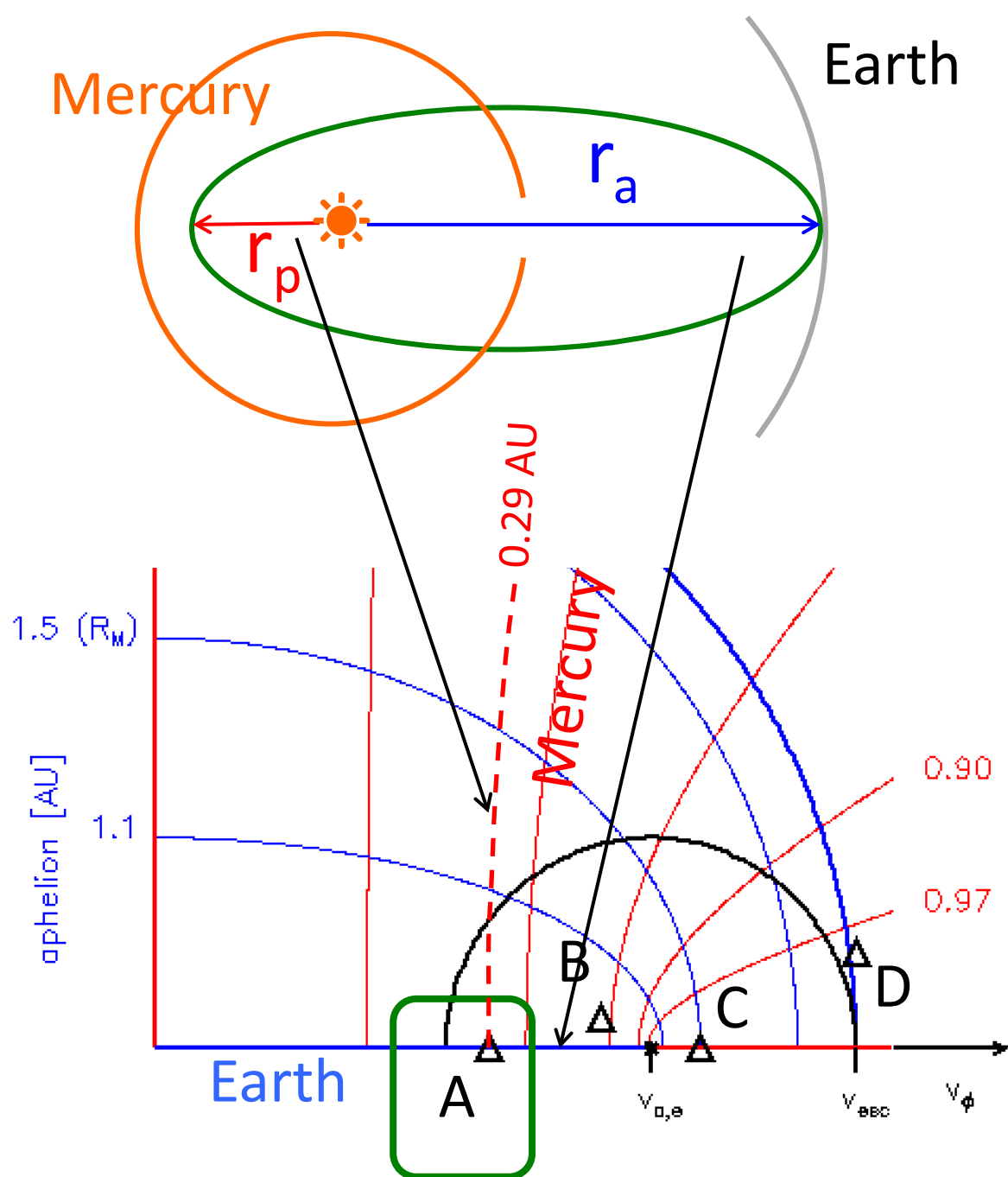
1976: sampled solar wind
 $0.29 < r < 1$ AU

☐ New Horizons:

Earth $\rightarrow$ Jupiter $\rightarrow$ out
 passing Pluto
 & Ultima Thule

☐ Cassini:

Earth $\rightarrow$ Venus $\rightarrow$ Earth
 $\rightarrow$ Venus $\rightarrow$ Saturn



☐ MAVEN:
Earth $\rightarrow$ Mars direct

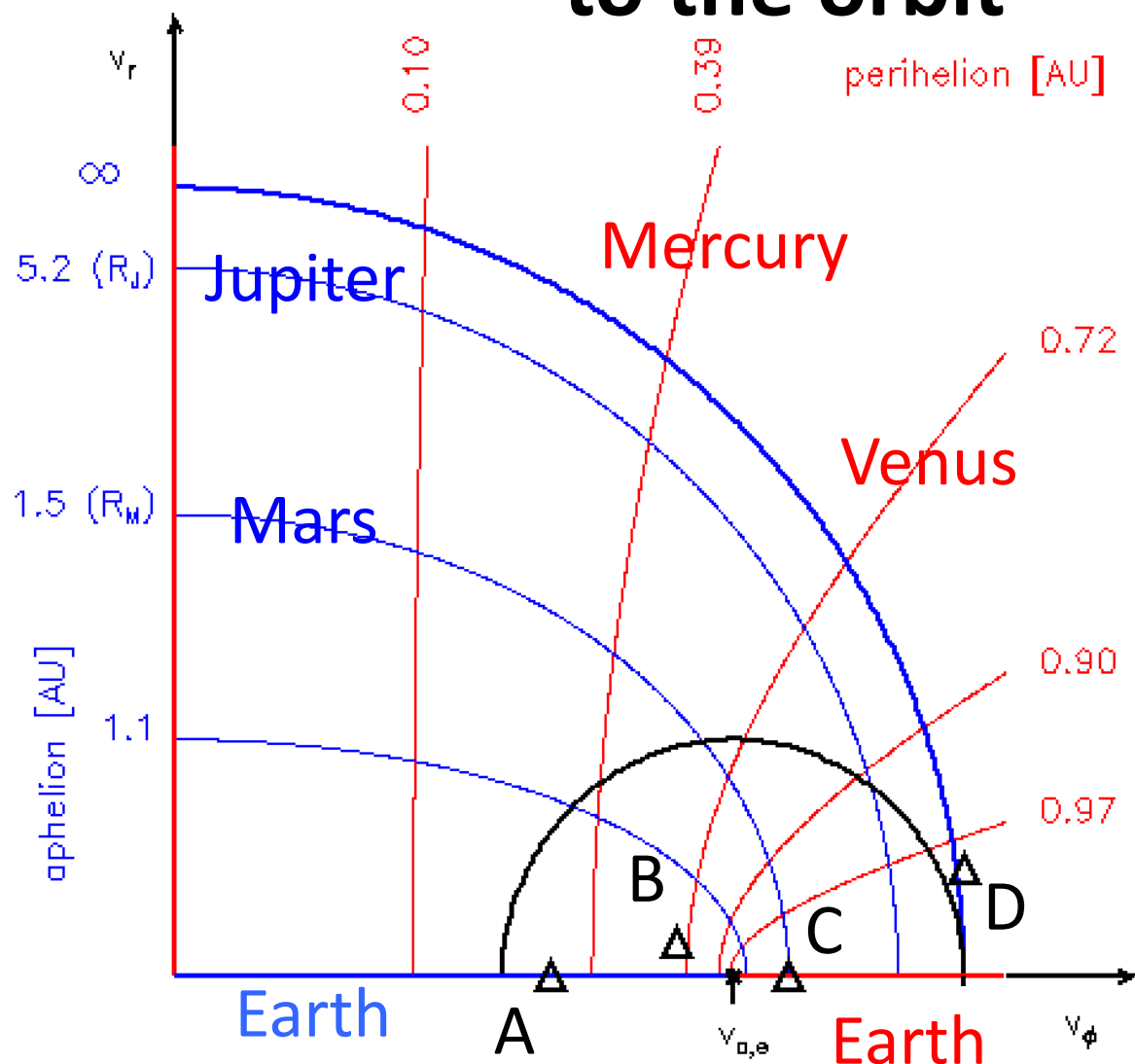
☒ Helios 2:
1976: sampled solar
wind $0.29 < r < 1$ AU

☐ New Horizons:
Earth $\rightarrow$ Jupiter $\rightarrow$ out
passing Pluto
& Ultima Thule

☐ Cassini:
Earth $\rightarrow$ Venus $\rightarrow$ Earth
 $\rightarrow$ Venus $\rightarrow$ Saturn

R=1 AU

Match the mission to the orbit



C MAVEN:
Earth $\rightarrow$ Mars direct

A Helios 2:
1976: sampled solar
wind $0.29 < r < 1$ AU

D New Horizons:
Earth $\rightarrow$ Jupiter $\rightarrow$ out
passing Pluto
& Ultima Thule

☐ Cassini:
Earth $\rightarrow$ Venus $\rightarrow$ Earth
 $\rightarrow$ Venus $\rightarrow$ Saturn

VENUS 1 FLYBY
26 APR 1998

VENUS 2 FLYBY
24 JUN 1999

VENUS
TARGETING
MANEUVER
3 DEC 1998

LAUNCH
15 OCT 1997

EARTH FLYBY
18 AUG 1999

SATURN ORBIT INSERTION
1 JUL 2004

JUPITER'S ORBIT
11.8 YEARS

JUPITER
FLYBY
30 DEC 2000

VEN:

→ Mars direct

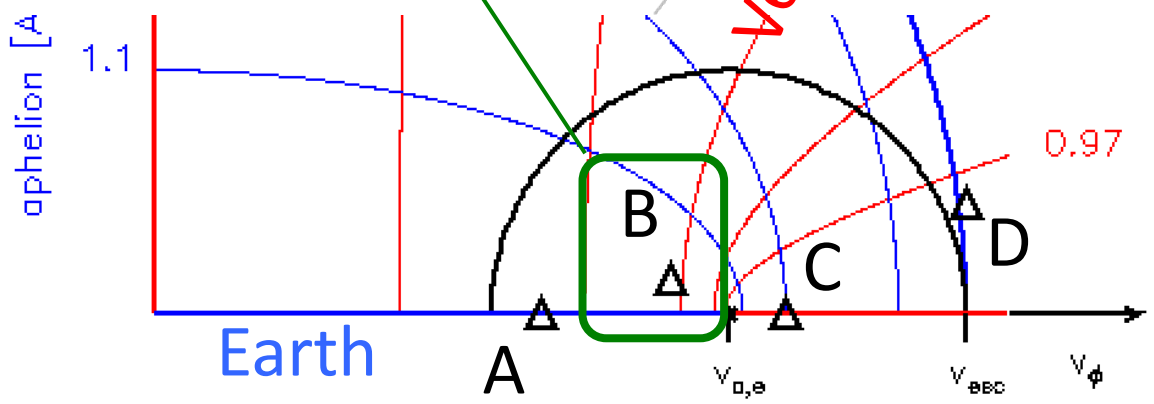
os 2:

sampled solar
 $0.29 < r < 1$ AU

v Horizons:

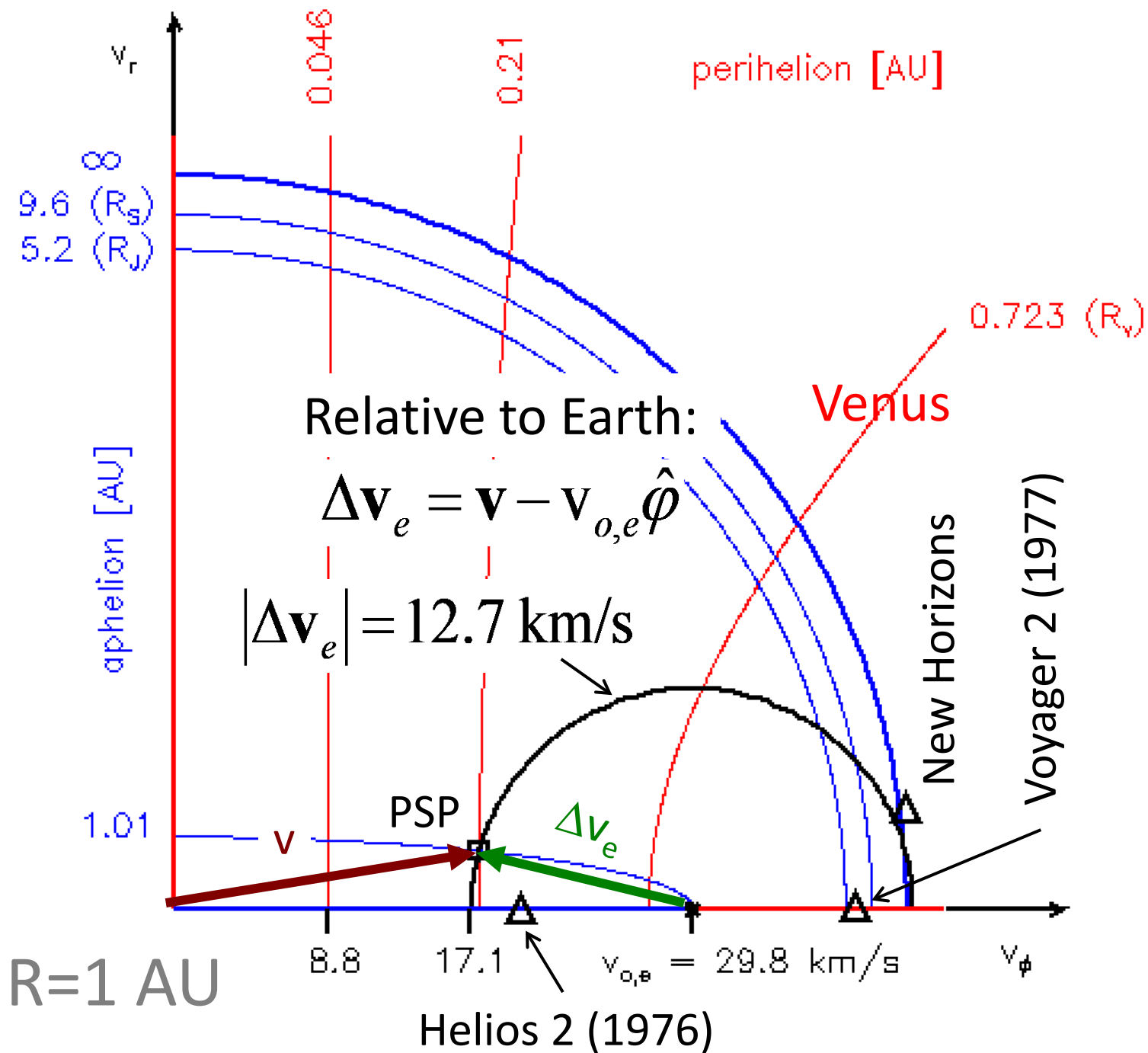
→ Jupiter → out

passing Pluto
& Ultima Thule

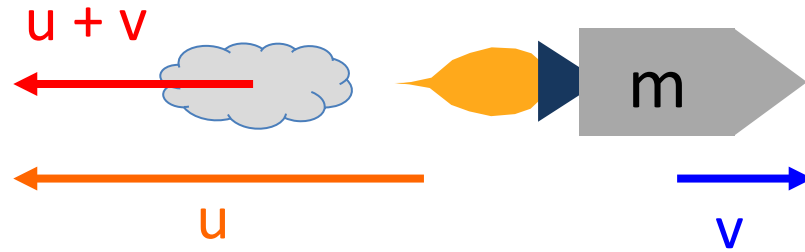


B Cassini:

Earth → Venus → Earth
→ Venus → Saturn



$$m \frac{dv}{dt} = F = |\dot{m}| u$$

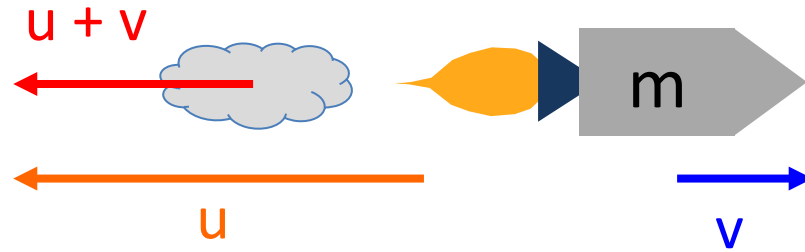


$$\Delta v = u \ln \left(\frac{|\Delta m_f| + m_p}{m_p} \right) \approx u \ln \left(\frac{|\Delta m_f|}{m_p} \right)$$

Impulse: $\int F dt = \int u \dot{m} dt = u |\Delta m|$

Specific impulse: $u = \frac{\int F dt}{|\Delta m|}$ [m/s] momentum/mass

$$m \frac{dv}{dt} = F = |\dot{m}| u$$

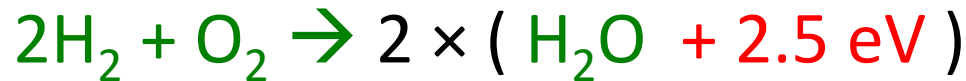


$$\Delta v \approx u \ln \left(\frac{|\Delta m_f|}{m_p} \right)$$

depends on:

1. mass ratio fuel : payload
2. specific impulse u

Specific impulse depends on fuel:



Enthalpy of reaction per mass of reactants →

$$u \simeq 4 \text{ km/s}$$

Fuel required:

$$\frac{|\Delta m_f|}{m_p} \approx \exp \left(\frac{\Delta v}{u} \right)$$

Fuel required: $\frac{|\Delta m_f|}{m_p} \approx \exp\left(\frac{\Delta v}{u}\right)$

$u = 4 \text{ km/s}$

1. Trip out of solar system:

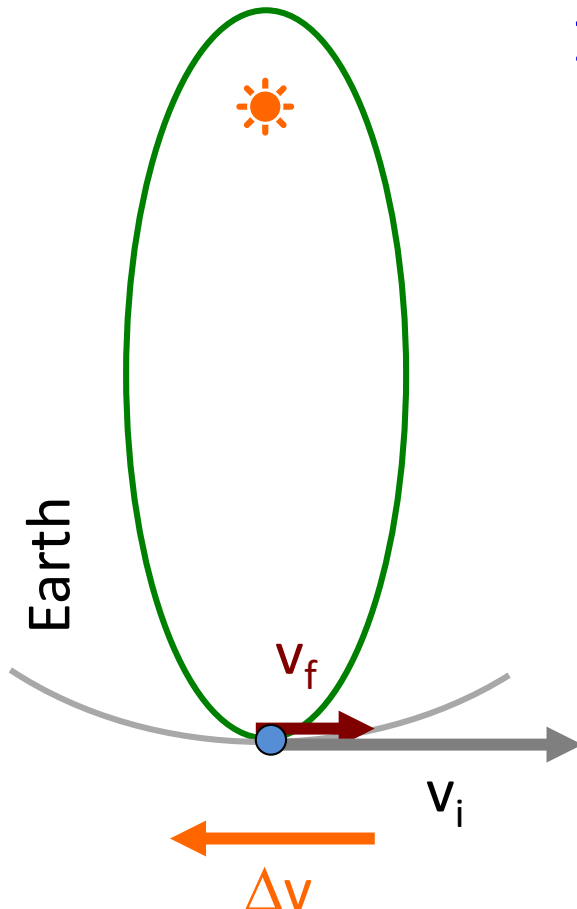
$\Delta v_e = 12 \text{ km/s}$

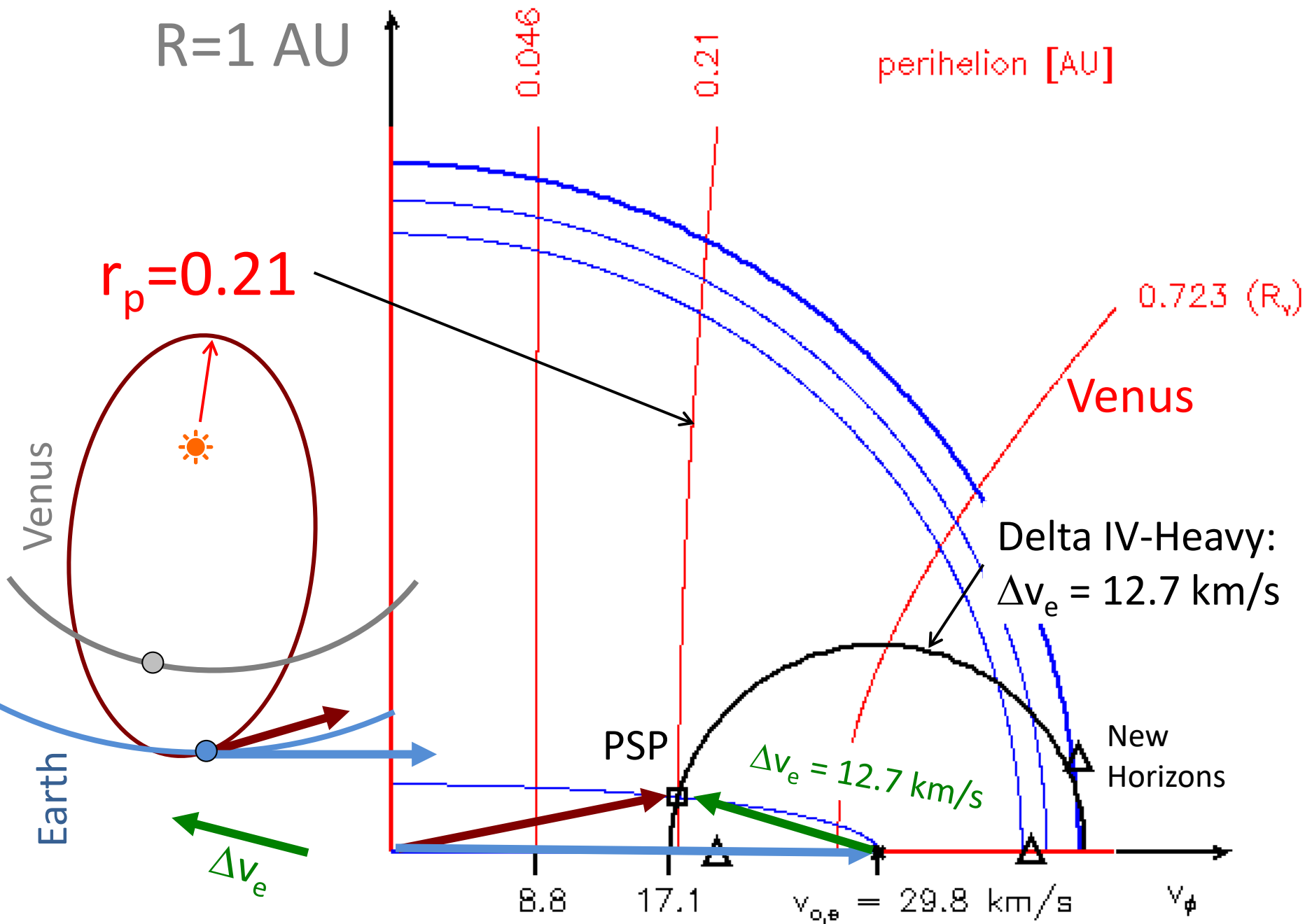
$$\frac{|\Delta m_f|}{m_p} = e^{12/4} = 20$$

2. Trip to Sun (0.045 AU)

$\Delta v_e = 21 \text{ km/s}$

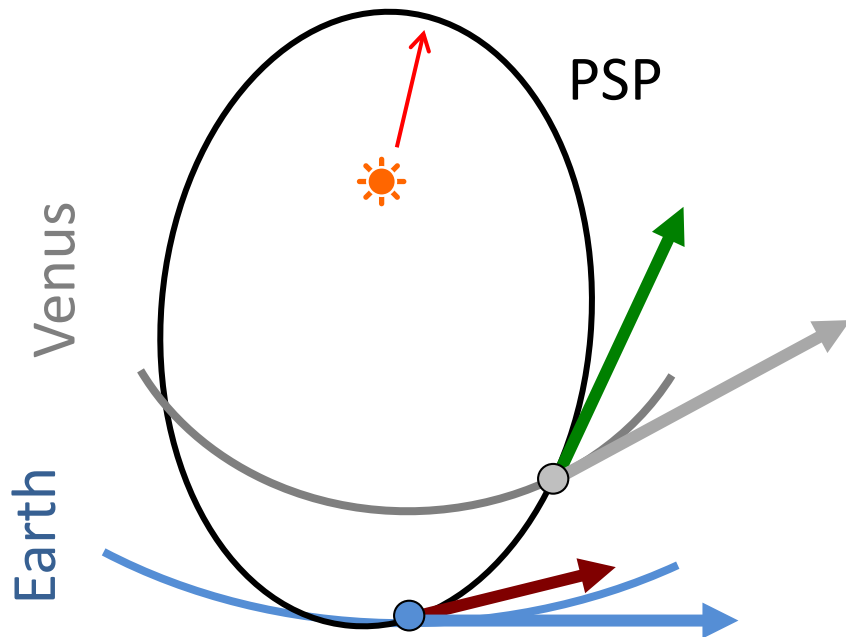
$$\frac{|\Delta m_f|}{m_p} = e^{21/4} = 190$$





Changing reference

$R=0.72$ AU



Earth

aphelion [AU]

32 km/s

PSP

Venus

perihelion [AU]

0.723 (R_V)

Venus

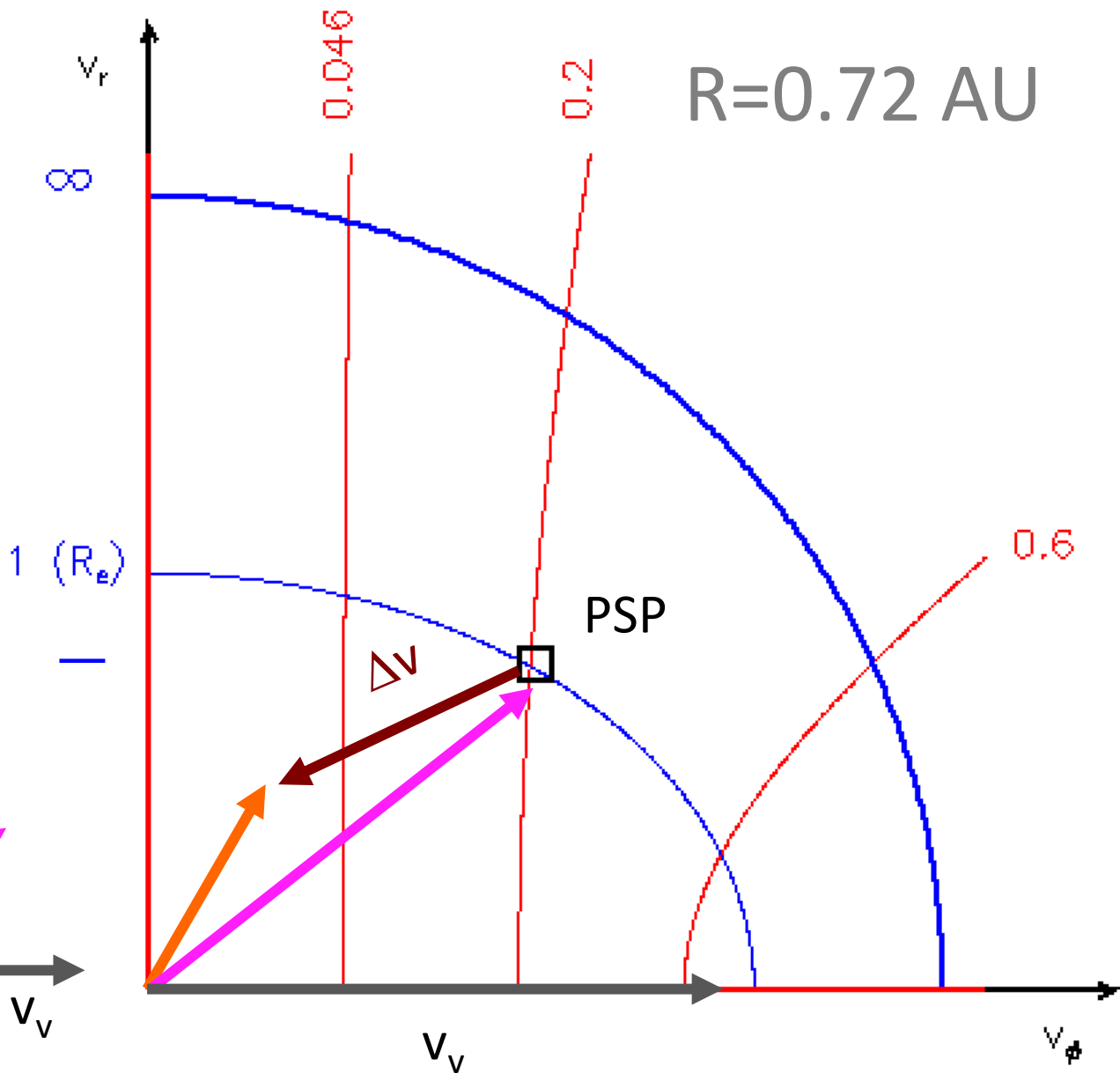
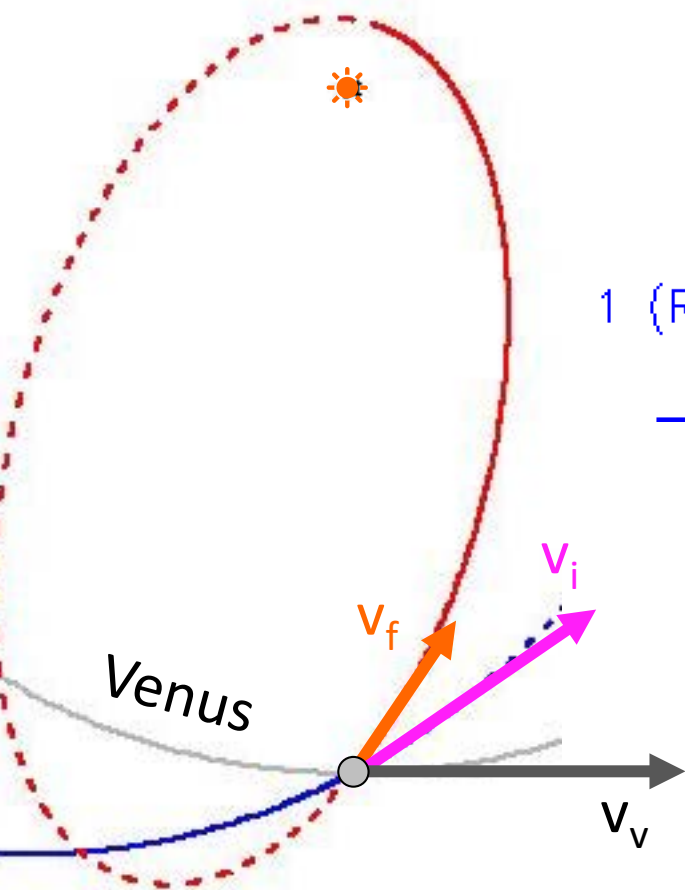
17 km/s

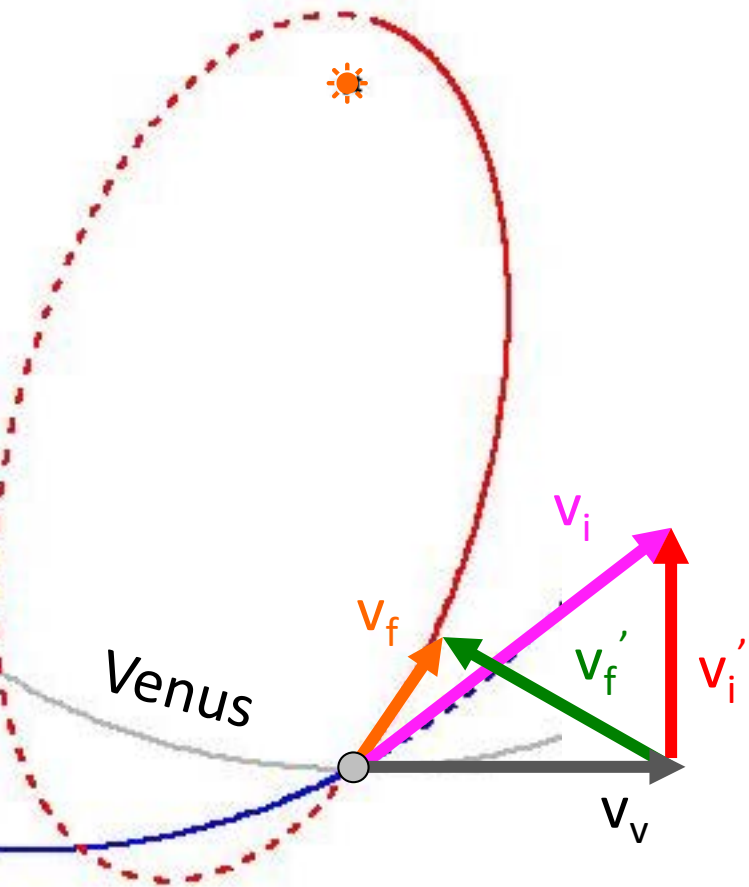
PSP

Earth

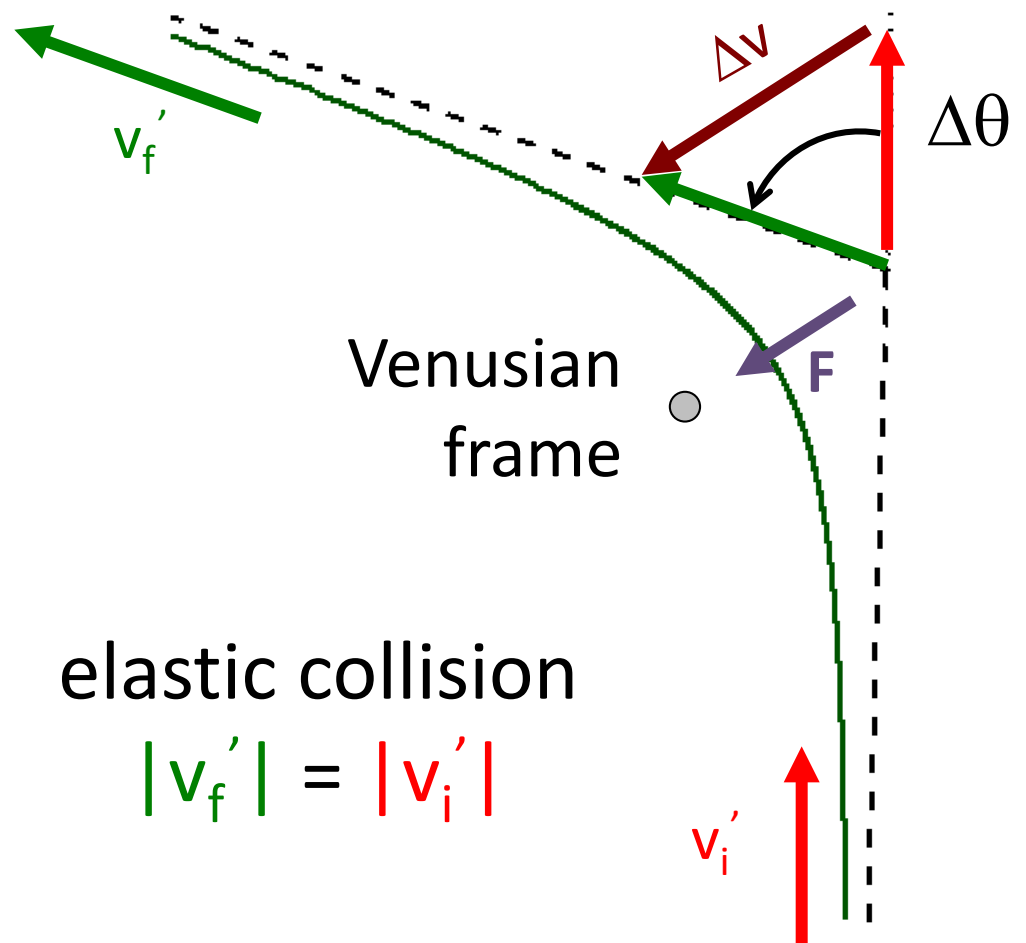
$R=1$ AU

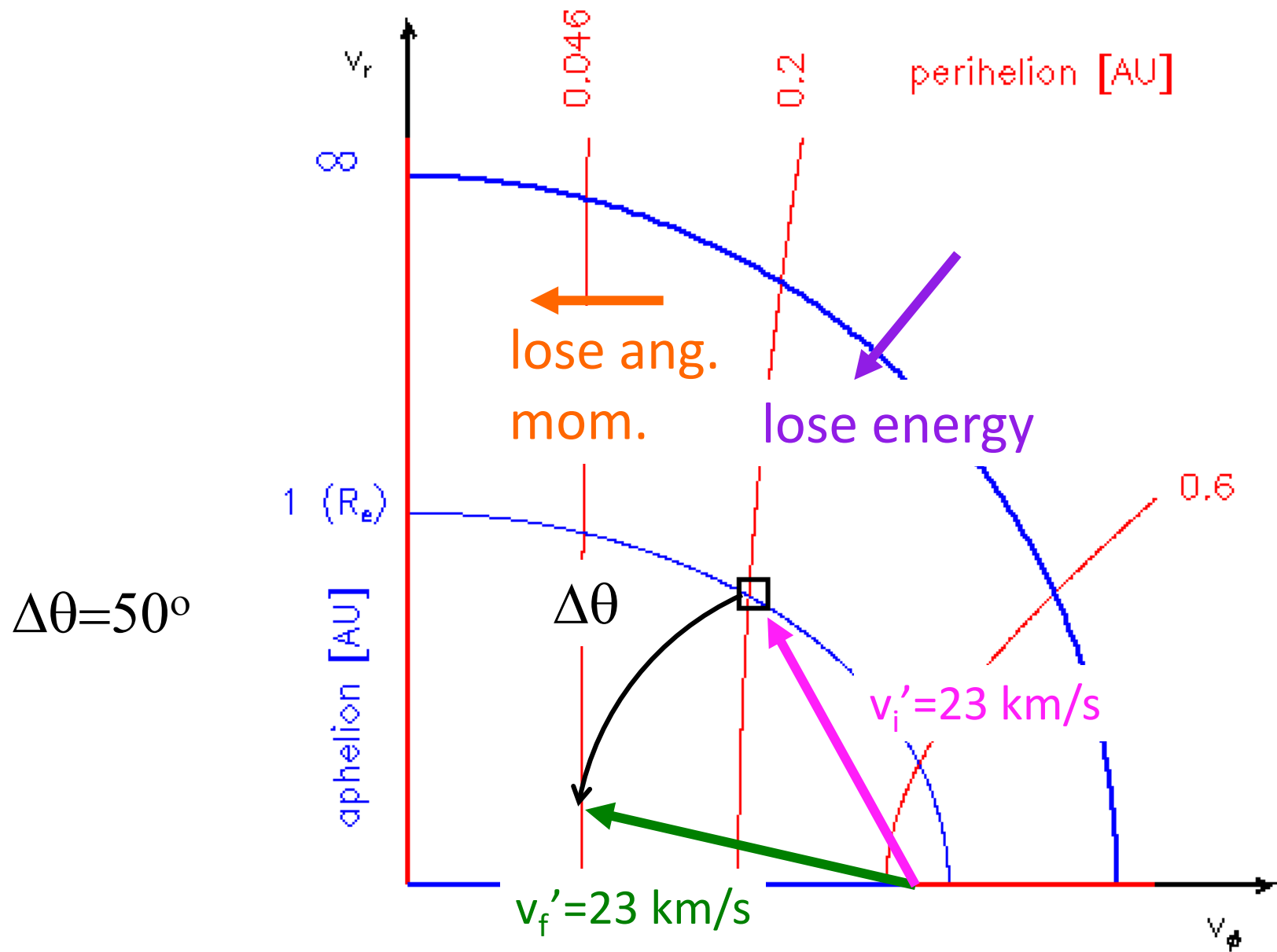
Δv for free:
Venus gravity
assist





Kepler
meets
Rutherford





$$\Delta\theta = 2\sin^{-1}\left(\frac{1}{1 + 2v_i'^2 / v_{e,p}^2}\right)$$

$$v_i' = 23 \text{ km/s}$$

$$\Delta\theta = 50^\circ$$

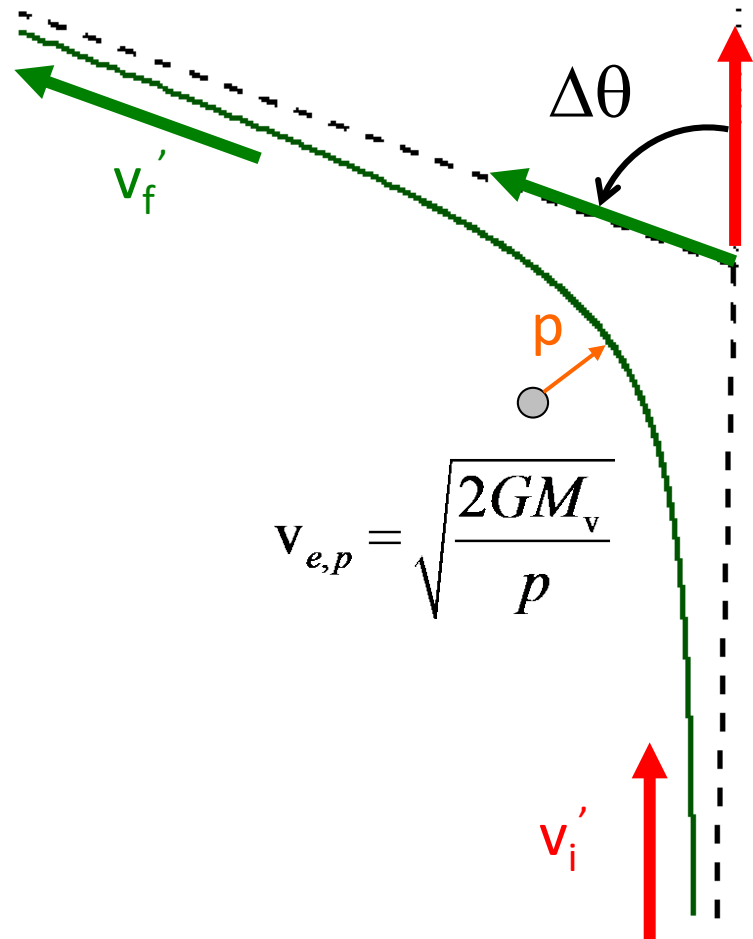
$$\Rightarrow v_{e,p} = 28 \text{ km/s}$$

$$\Rightarrow p = 770 \text{ km}$$

surface of Venus @

$$R_v = 6,000 \text{ km}$$

Rutherford
scattering angle



$$\Delta\theta = 2\sin^{-1}\left(\frac{1}{1 + 2v_i'^2 / v_{e,p}^2}\right)$$

$$v_i' = 23 \text{ km/s}$$

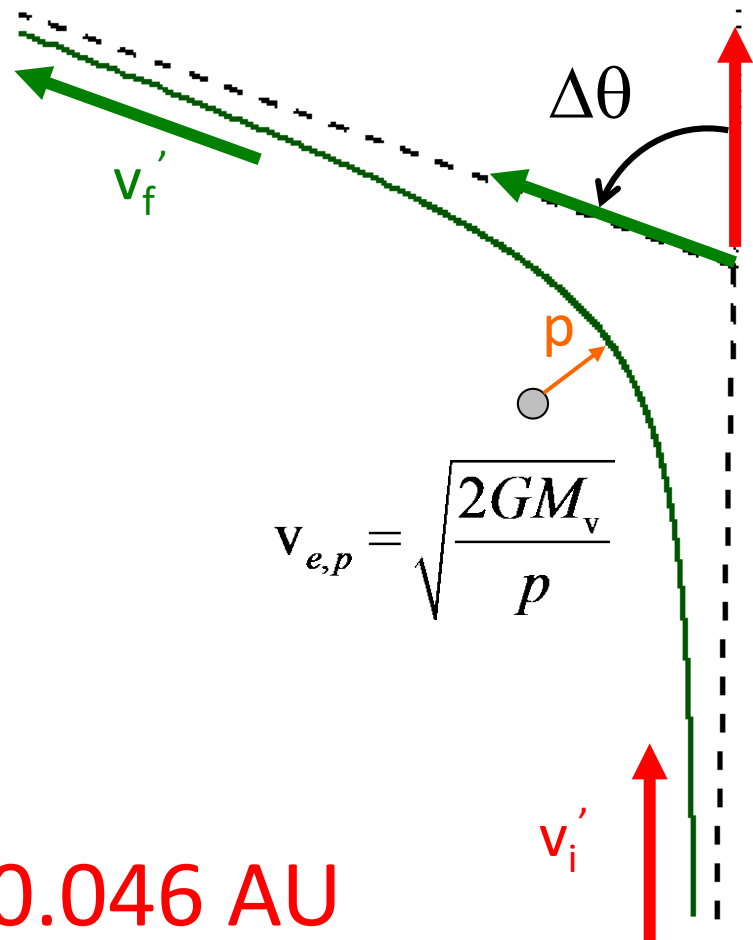
$$p > R_v = 6,000 \text{ km}$$

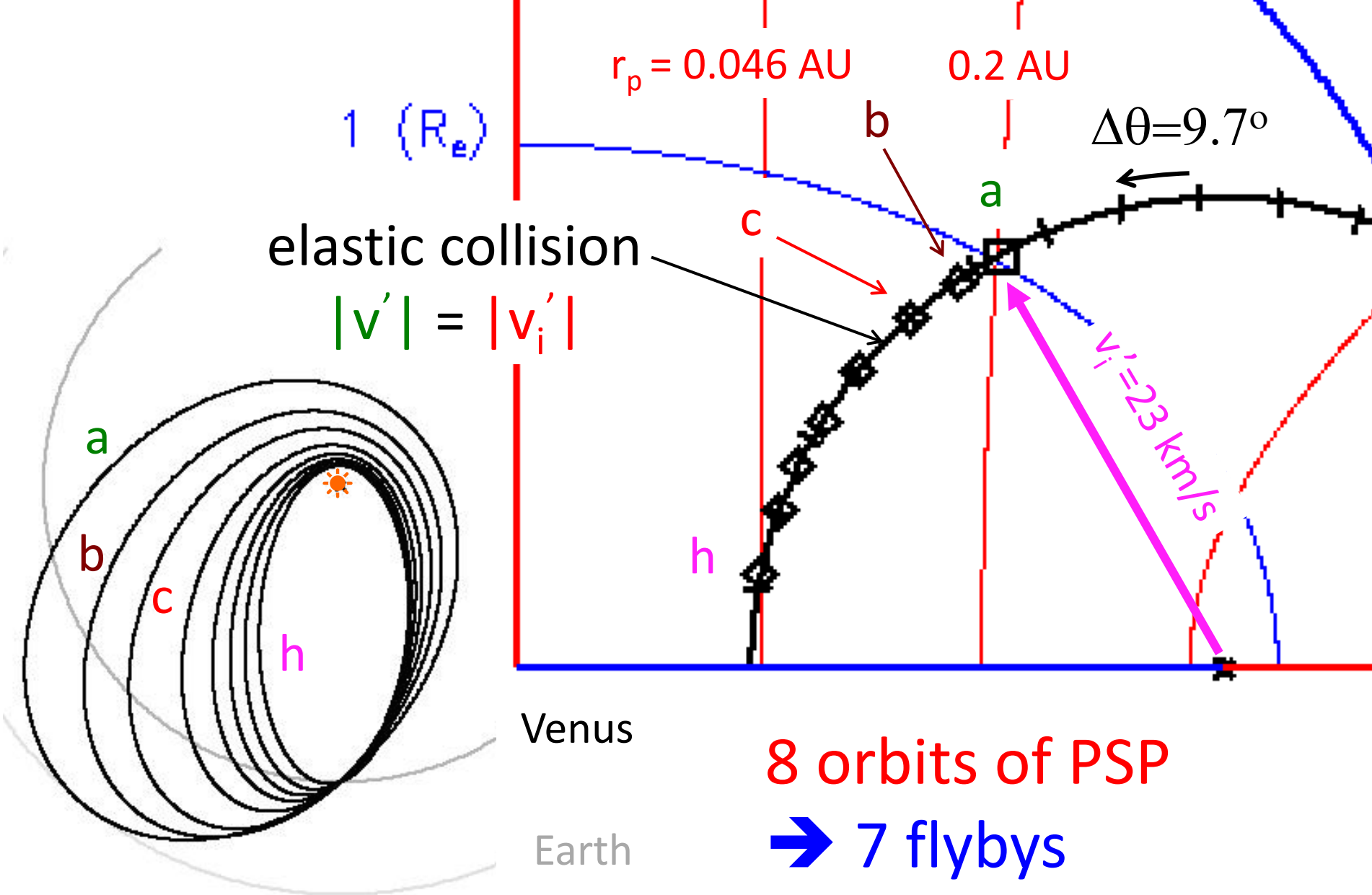
$$v_{e,p} < 10.0 \text{ km/s}$$

$$\Rightarrow \Delta\theta < 9.7^\circ$$

$$50^\circ > 5 \Delta\theta$$

$$\Rightarrow 6+ \text{ flybys for } r_p = 0.046 \text{ AU}$$





Why not just 6? Why the crazy spacing?

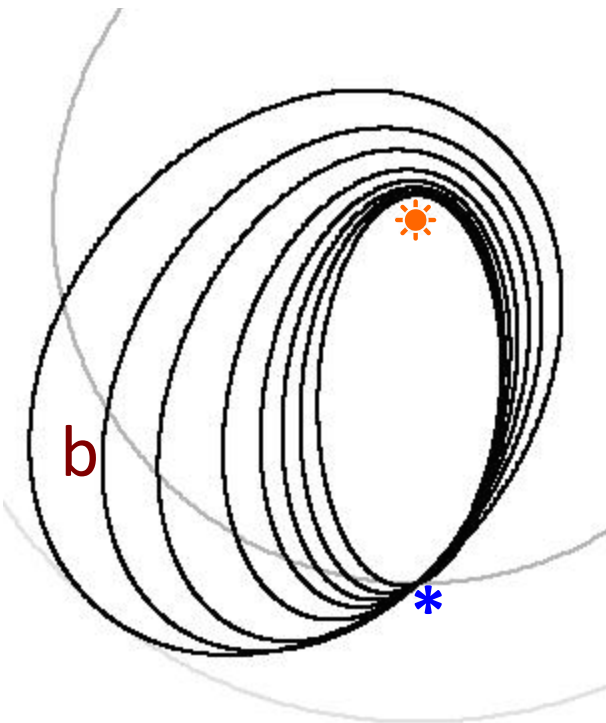
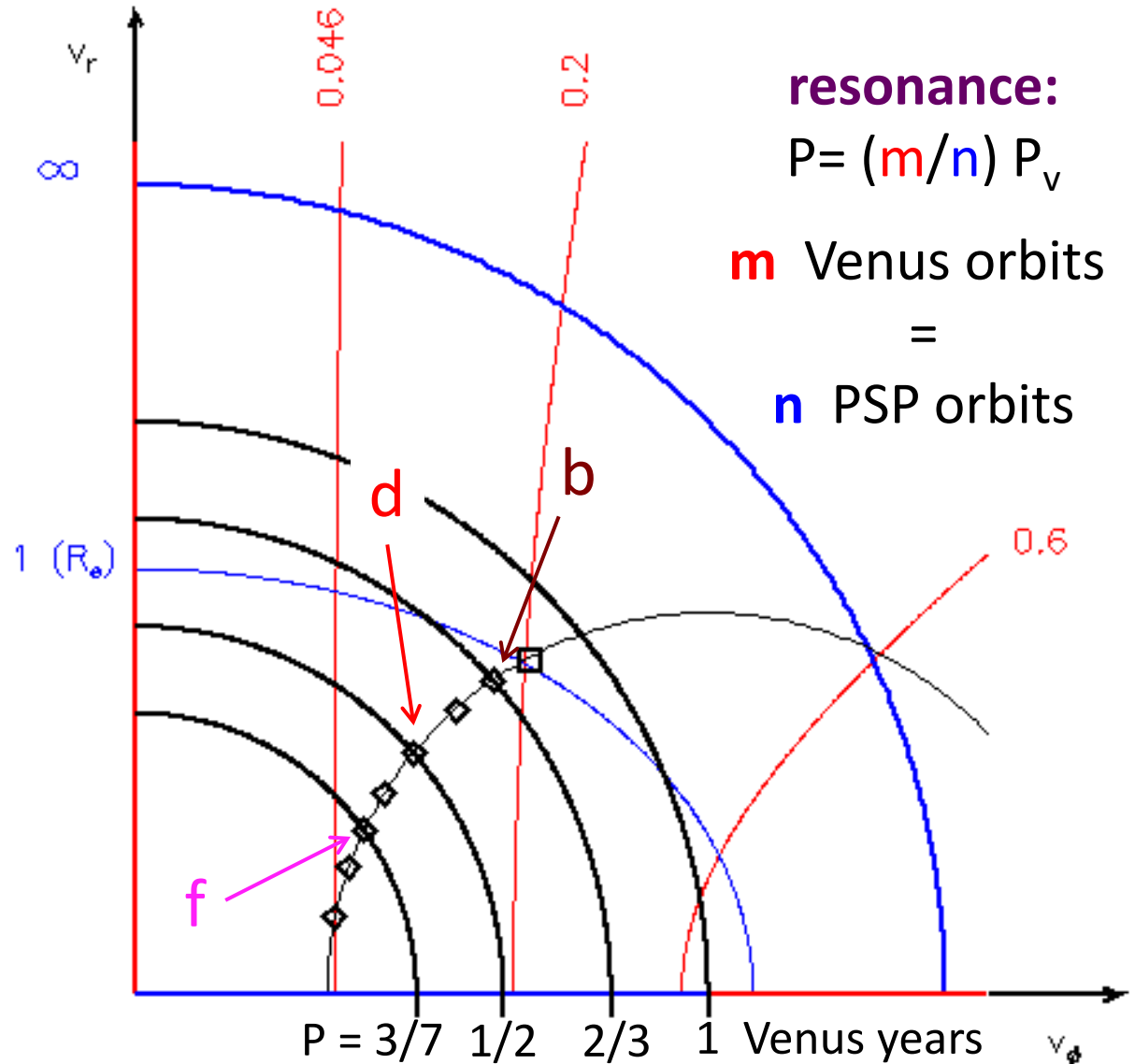
Kepler 3rd: orbital period

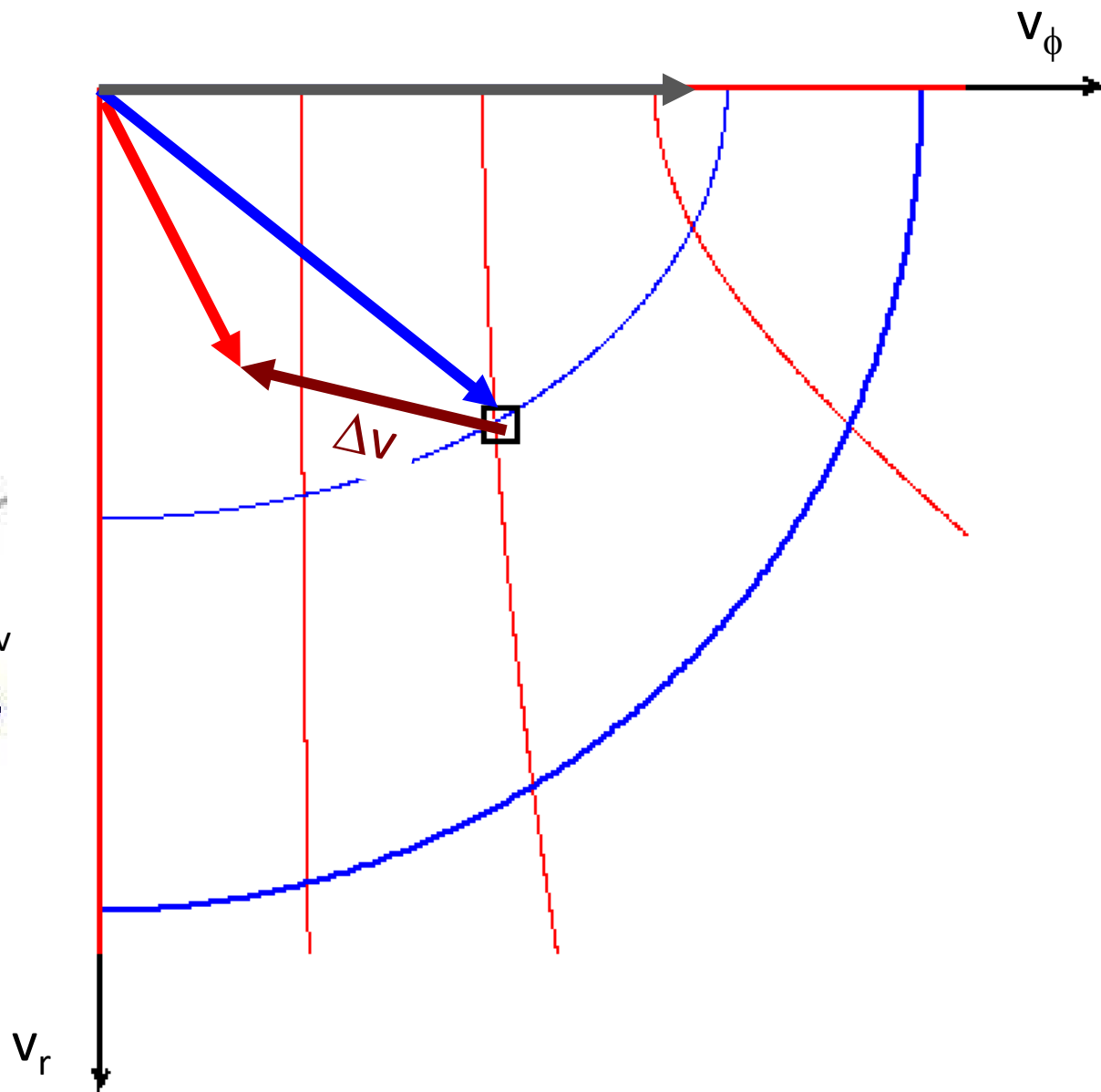
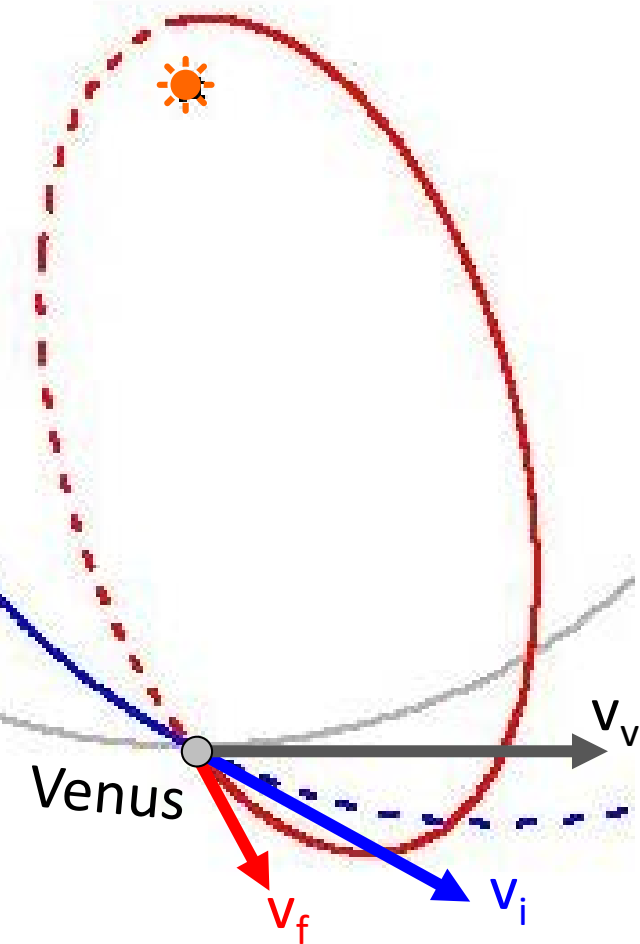
$$P \propto a^{3/2} \propto \left(2 - \mathbf{v}^2 / \mathbf{v}_o^2\right)^{-3/2}$$

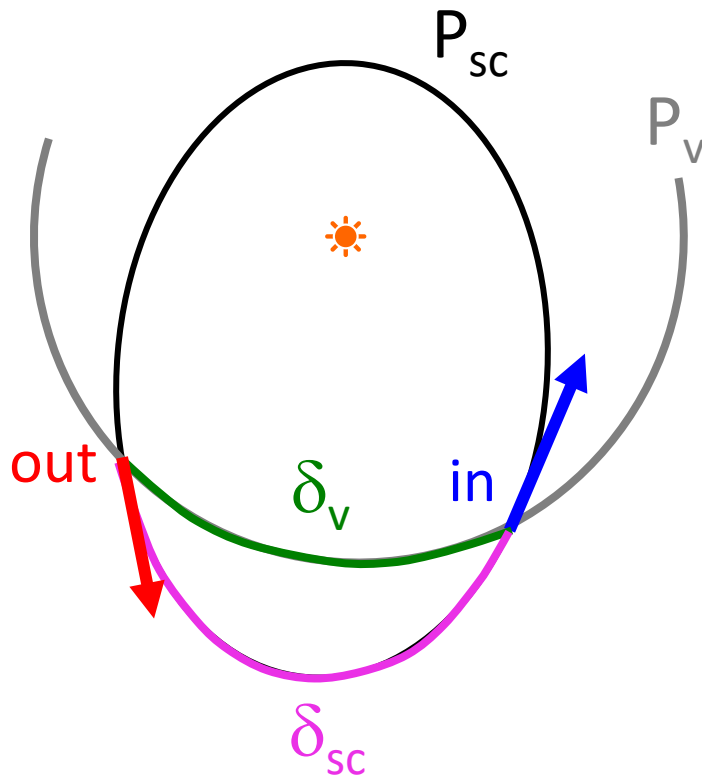
orbit **b**: $P = \frac{2}{3} P_v$
2 orbits by Venus
3 orbits by PSP
 in 450 days
 return to *

resonance:

$$P = (m/n) P_v$$

$$\frac{m}{n} = \frac{\text{Venus orbits}}{\text{PSP orbits}}$$






in-in or out-out
rendezvous

→ **m : n resonance**

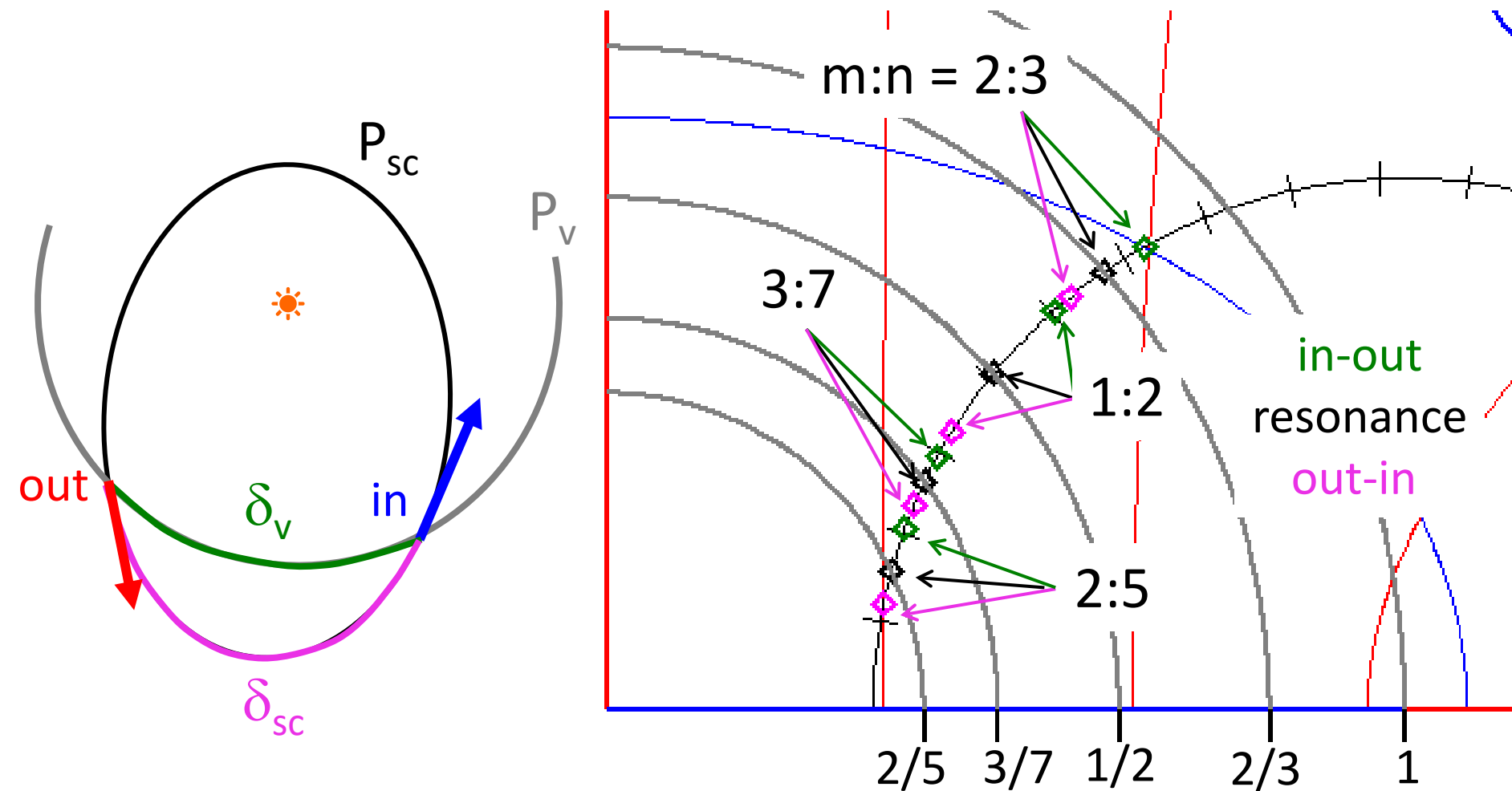
$$m P_v = n P_{sc}$$

in-out rendezvous:

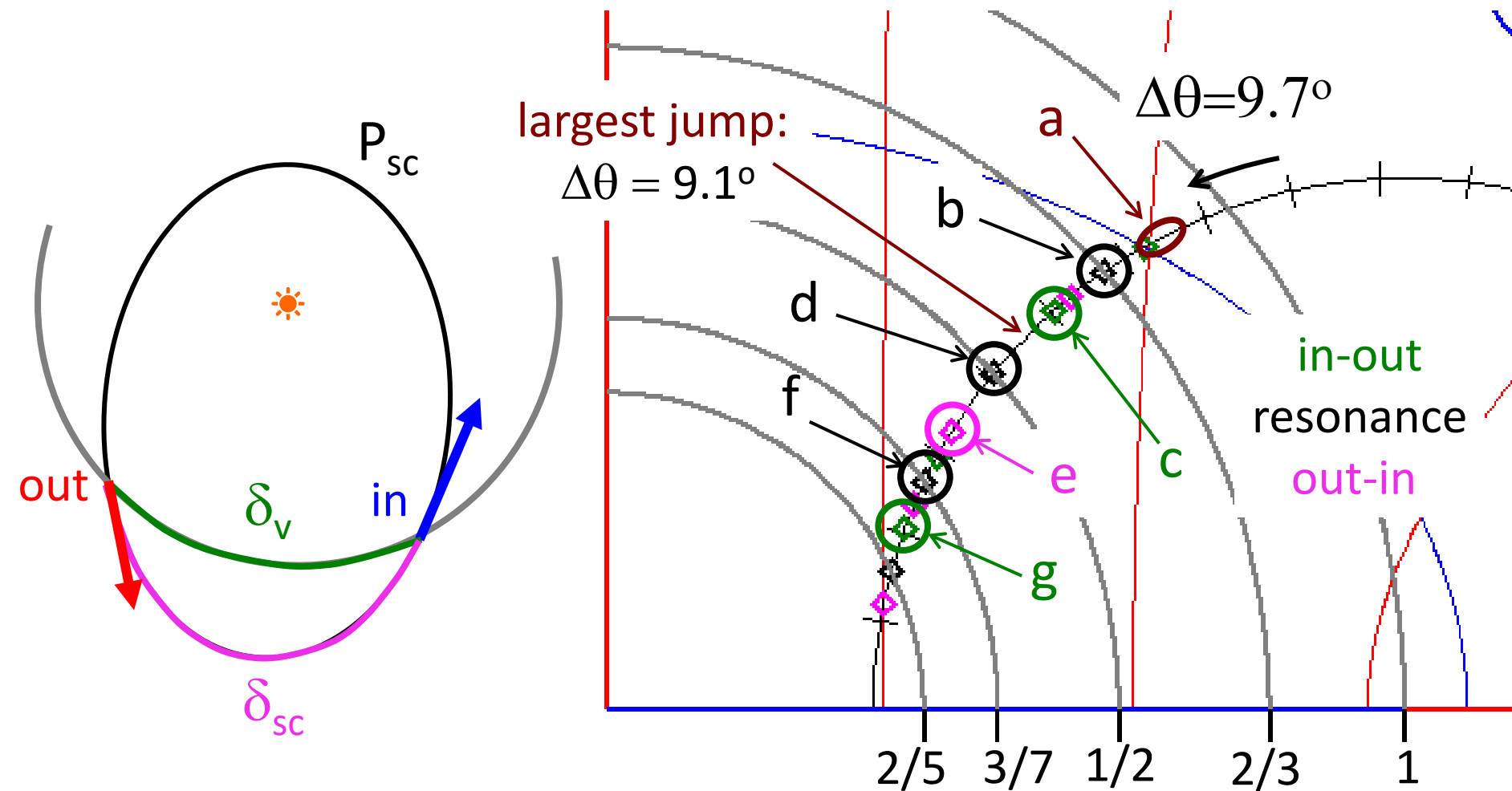
$$m P_v - \delta_v = n P_{sc} - \delta_{sc}$$

out-in rendezvous:

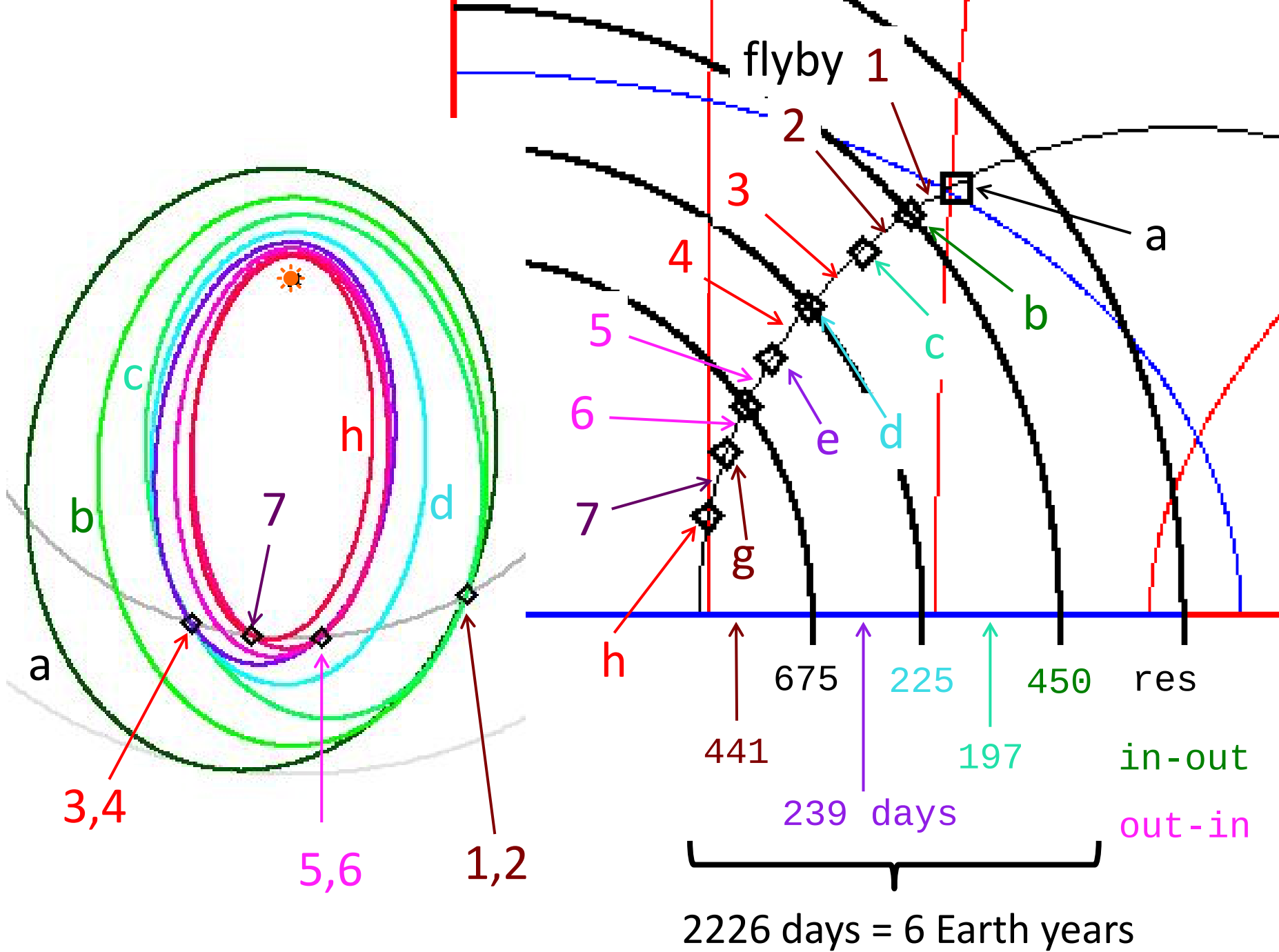
$$m P_v + \delta_v = n P_{sc} + \delta_{sc}$$



orbits allowing repeated flybys
after $\Delta t \leq 3 P_v$



legal moves: $\left\{ \begin{array}{l} \text{in-out} \rightarrow \text{out-in} \text{ [via out-out (res)]} \\ \text{out-in} \rightarrow \text{in-out} \text{ [via in-in (res)]} \end{array} \right.$



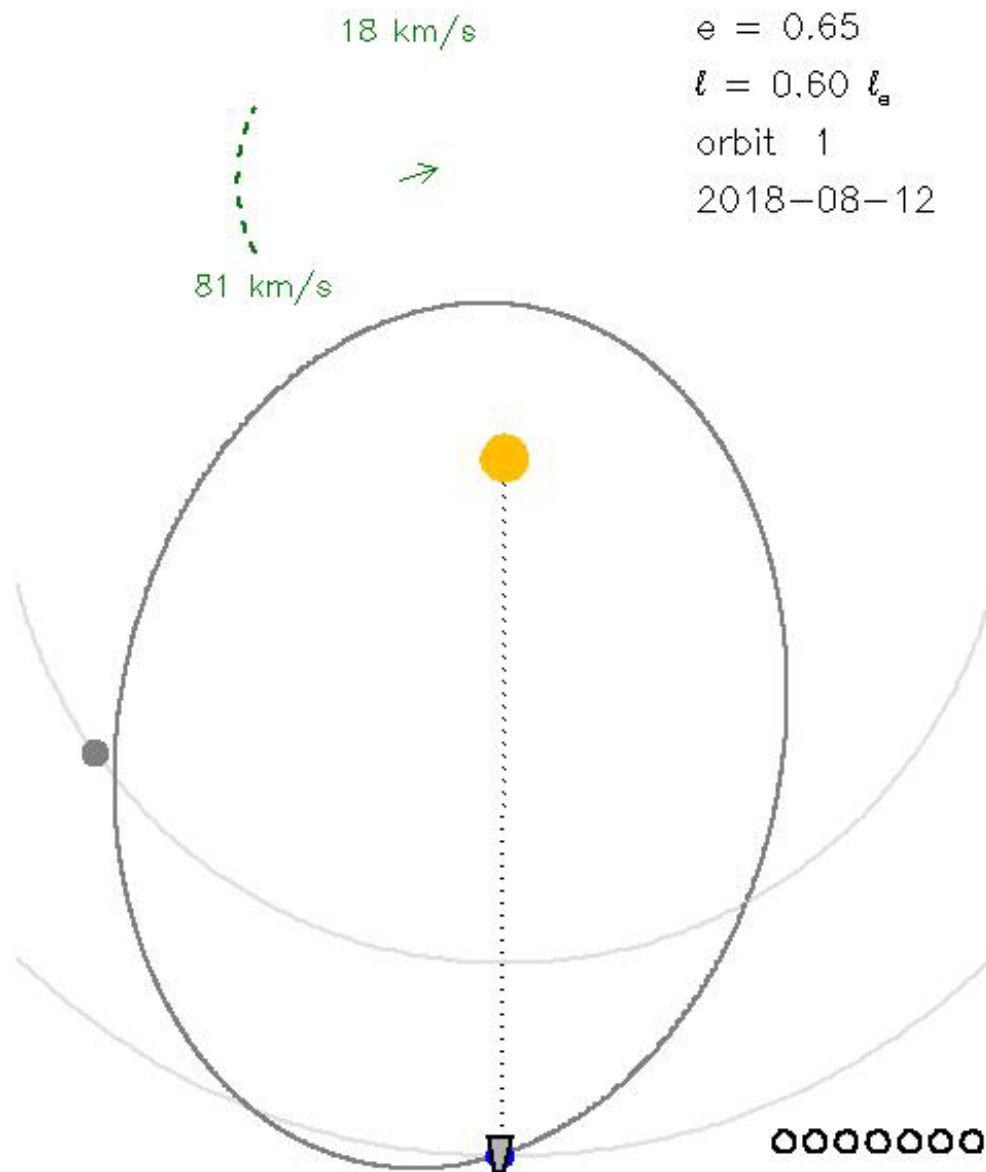
day 0.26

$e = 0.65$

$\ell = 0.60 \ell_{\oplus}$

orbit 1

2018-08-12

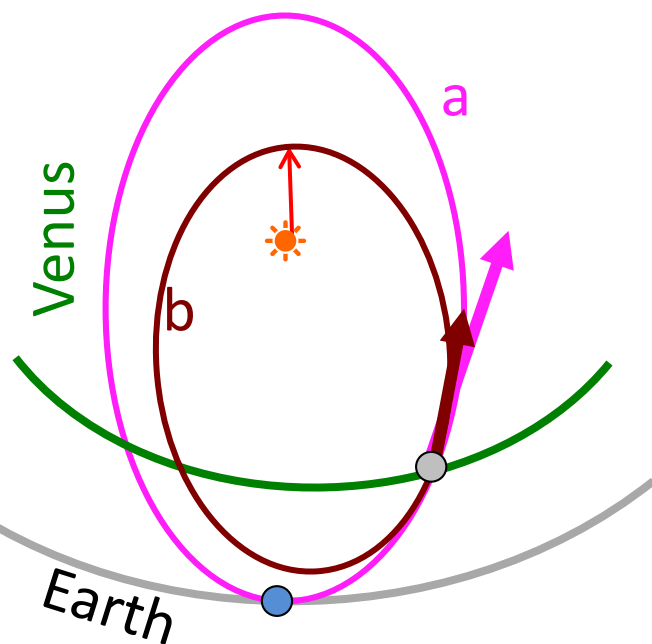


destination:

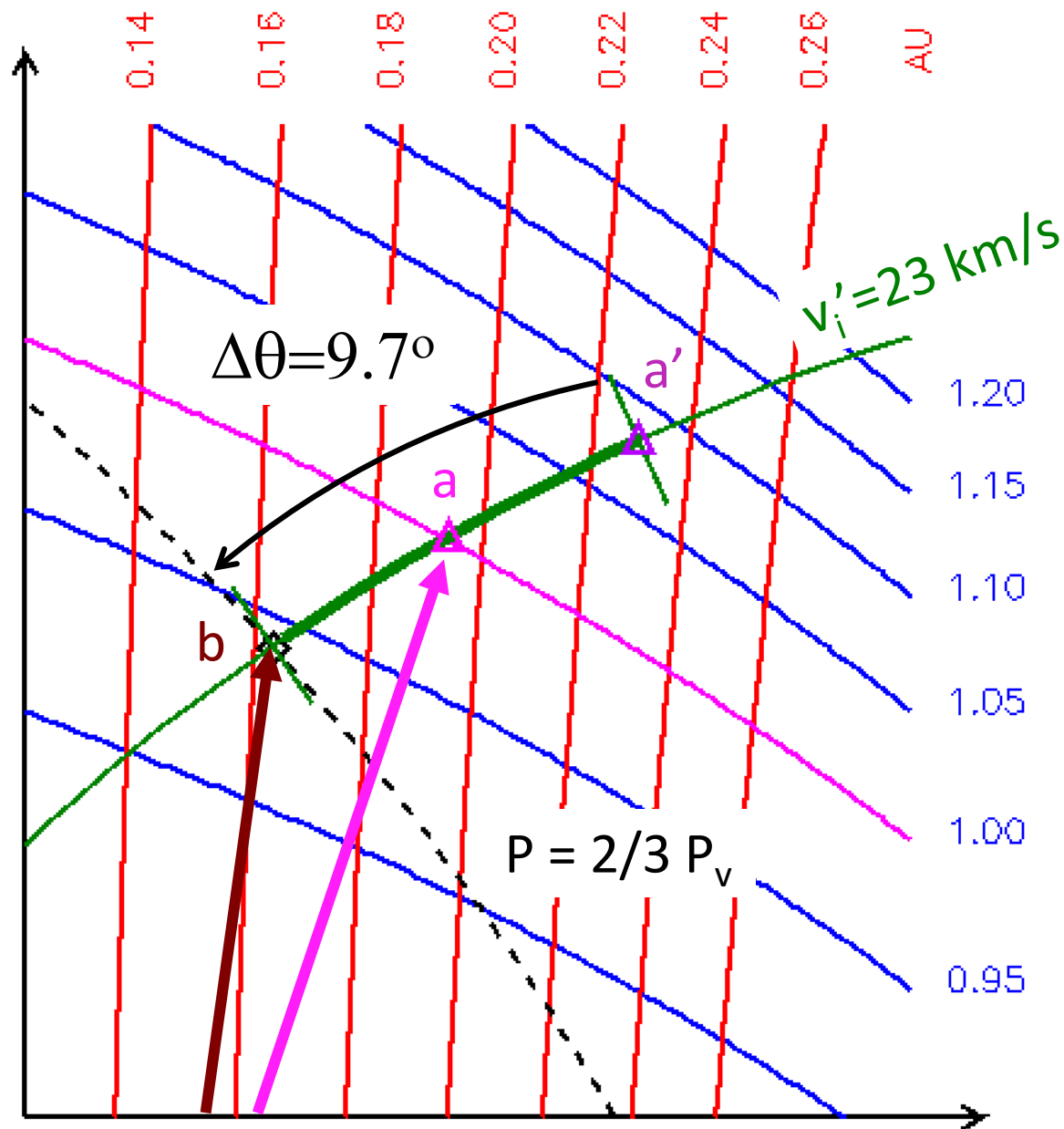
orbit b

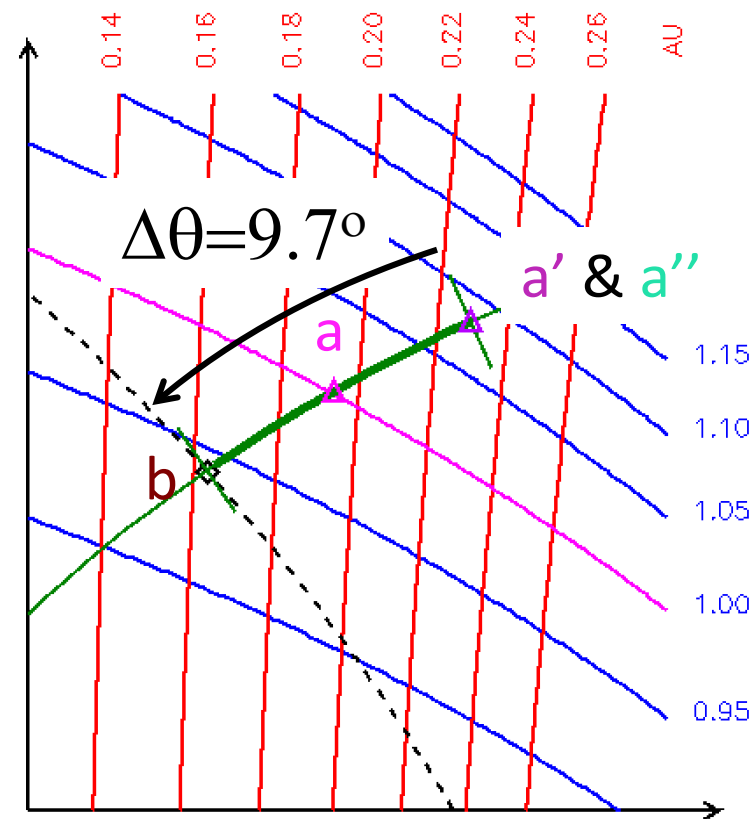
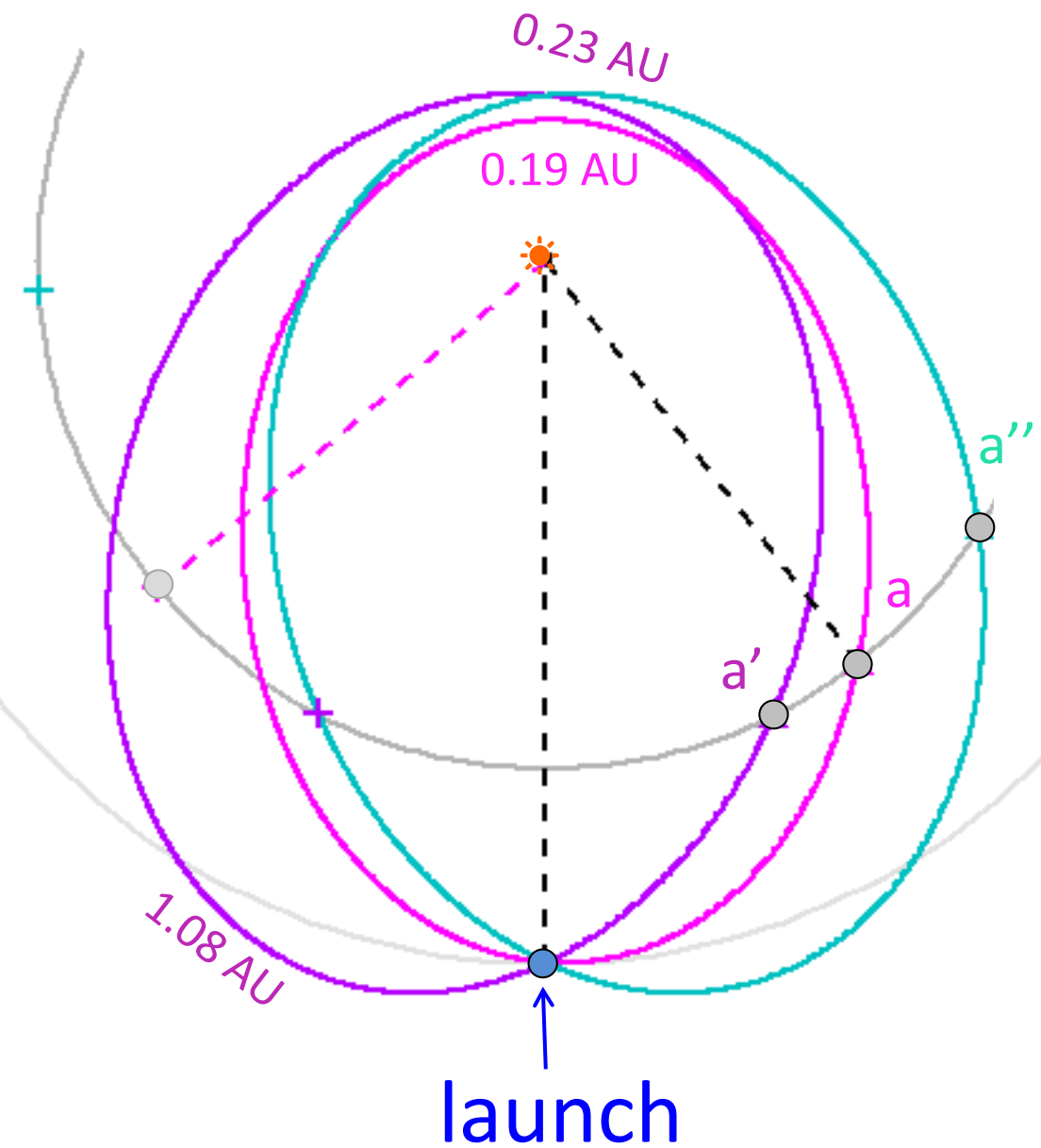
$r_a = 0.94$ AU

$r_p = 0.16$ AU



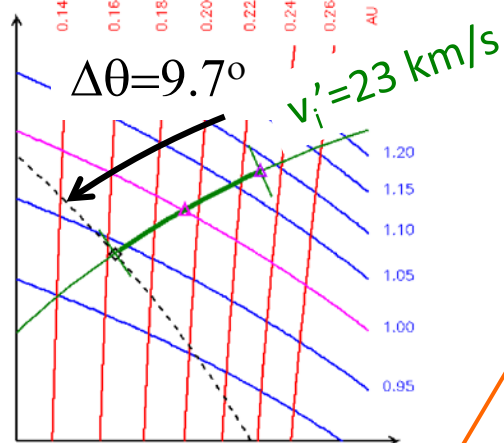
1st orbit: a





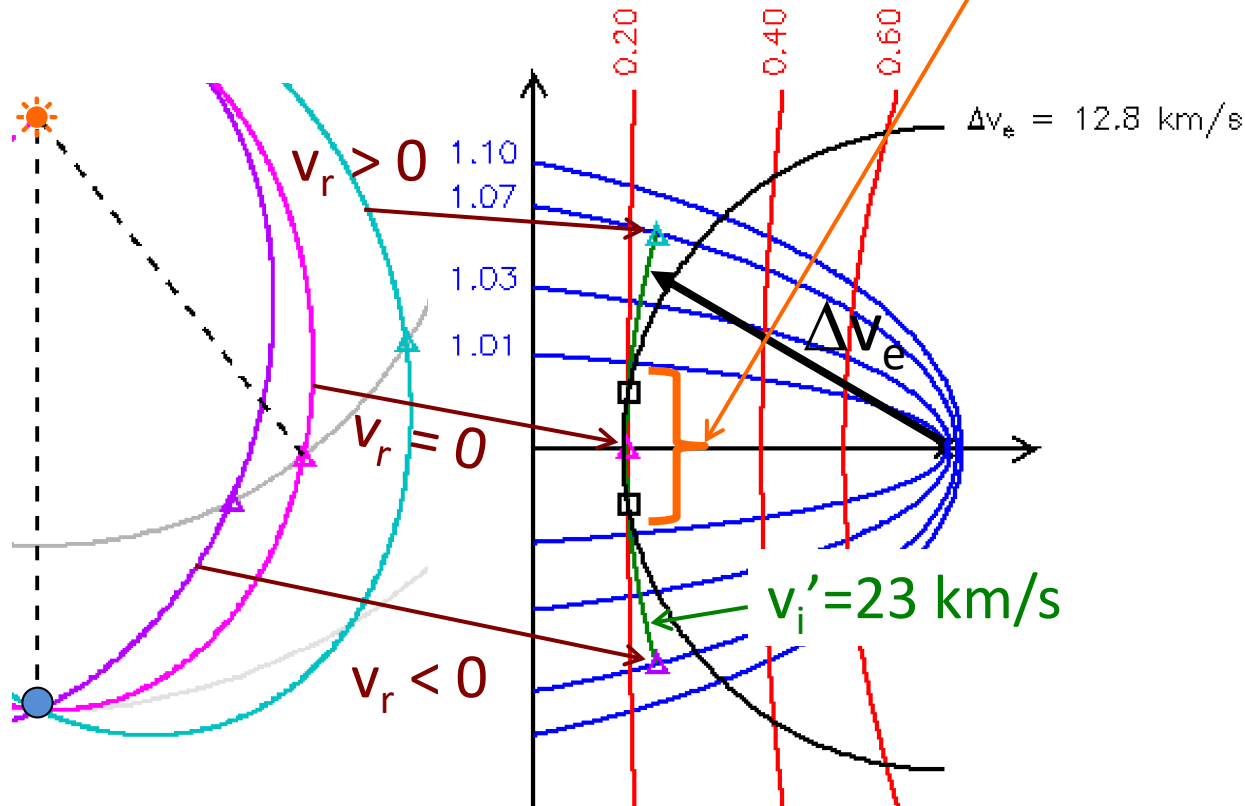
1st orbit: a

$R = 0.72 \text{ AU}$



Launch criteria:

- $\Delta v_e < 12.8 \text{ km/s}$
- $v_i' = 23 \text{ km/s}$
- $\Delta\theta < 9.7^\circ$



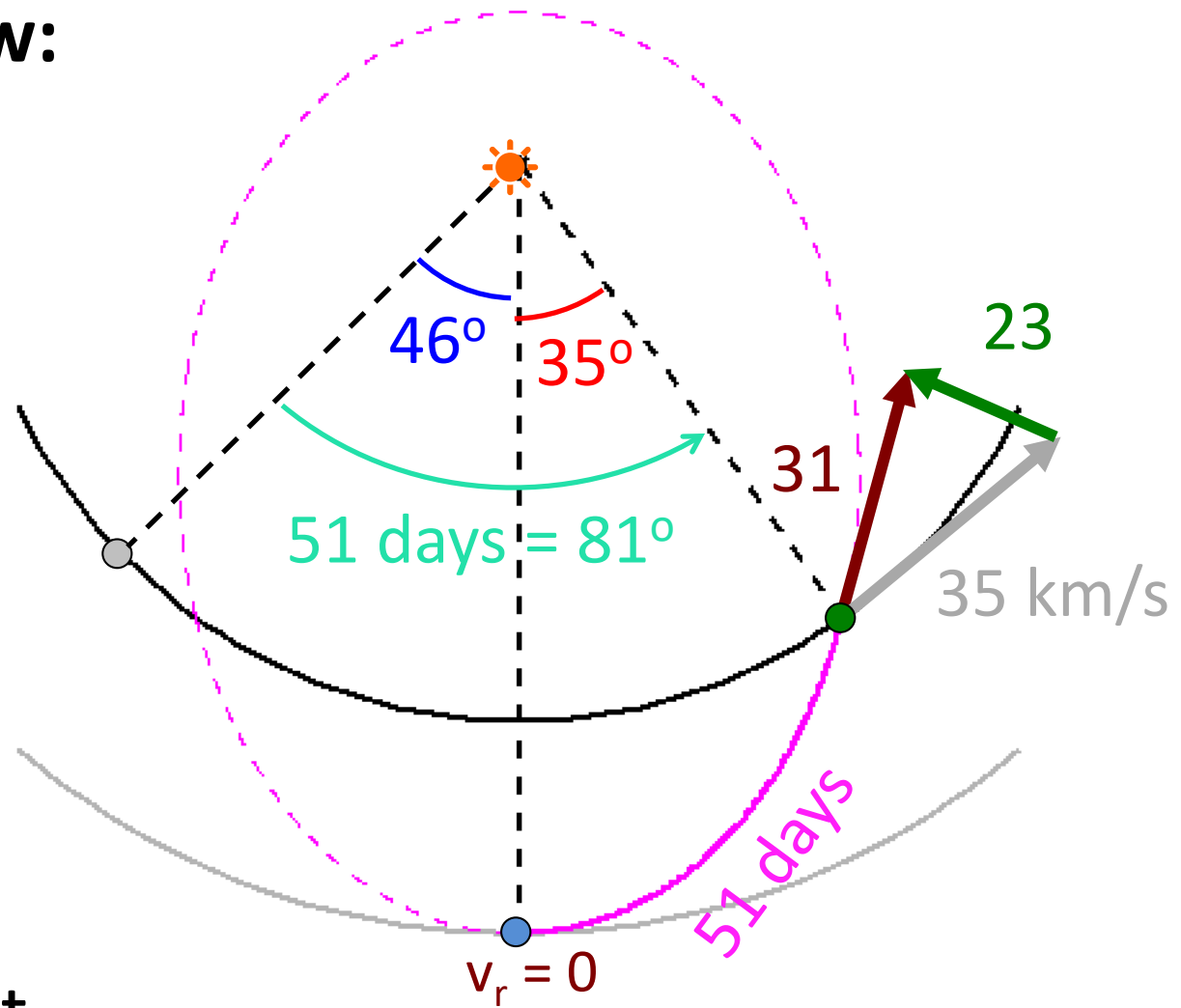
$R = 1 \text{ AU}$

Center of the Launch window:

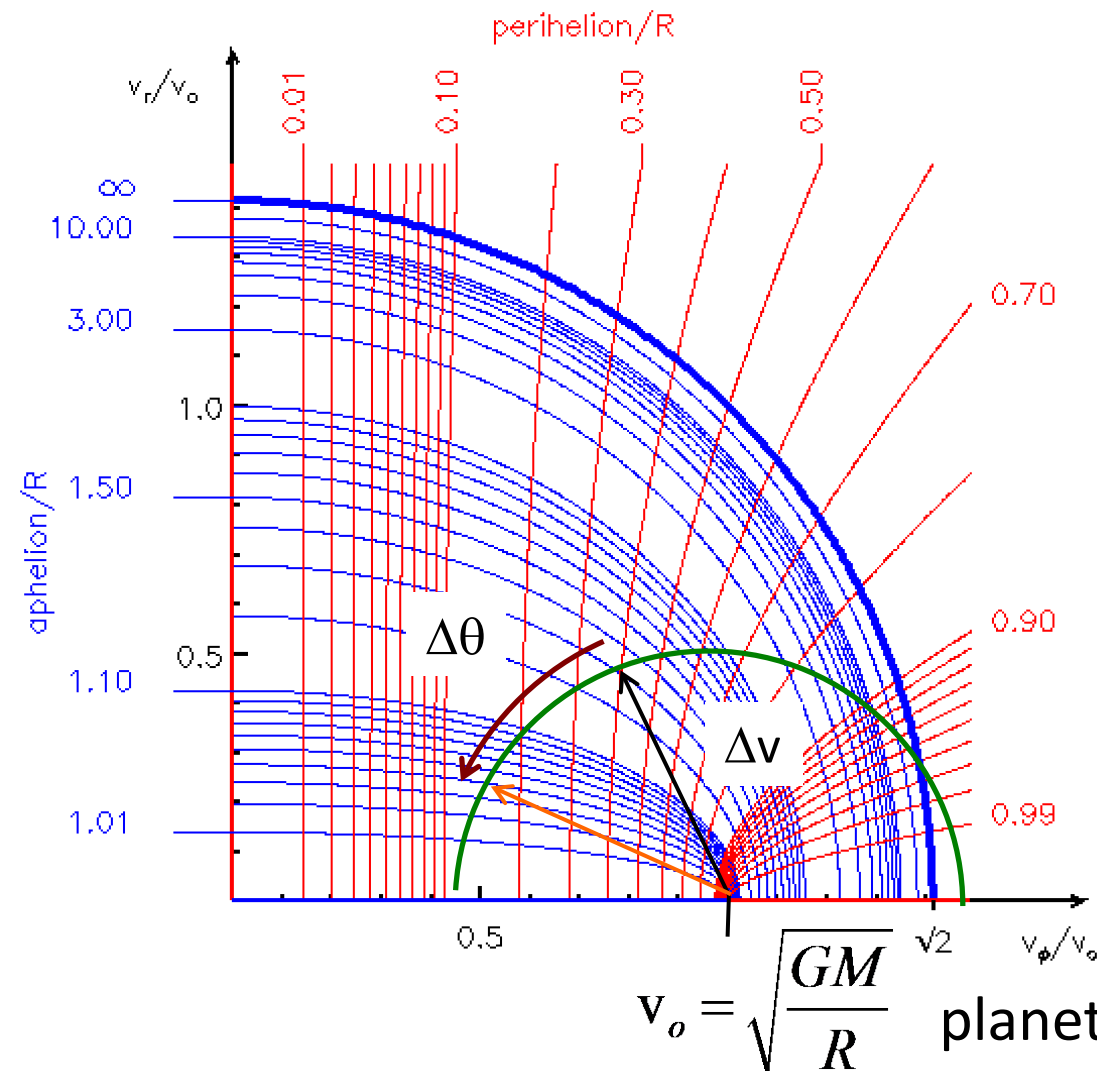
Venus lags
Earth by 46° on:

- 2015-06-05
- 2017-01-09
- **2018-08-12**
- 2020-03-20

19 months apart



Which Planets are Good?



$$\Delta\theta = 2\sin^{-1}\left(\frac{1}{1+2v_i'^2/v_{e,p}^2}\right)$$

$$v_i' = \Delta v \sim v_o$$

characteristic scattering angle

$$\Delta\theta_{ch} = 2\sin^{-1}\left[\frac{1}{1+2(v_o/v_{e,s})^2}\right]$$

planetary orbital speed

surface
escape
speed

Which Planets are Good?

characteristic scattering angle

$$\Delta\theta_{ch} = 2 \sin^{-1} \left[\frac{1}{1 + 2 \left(v_{e,s} / v_o \right)^{-2}} \right]$$

	Mercury	Venus	Earth	Mars	Jupiter
V_o [km/s]	48	35	30	24	13
$V_{esc,s}$ [km/s]	4.2	10.4	11.2	5.0	60
$V_{esc,s} / V_o$	0.087	0.28	0.37	0.21	4.6
$\Delta\theta_{ch}$	0.4°	4.5°	7.2°	2.4°	132°

Summary

- Launch speed (rel. to Earth) $\Delta \mathbf{v}_e$ limited by rocket
- Can get extra $\Delta \mathbf{v}$ from gravity assist: scattering off moving scatterer (planet)
- Elastic scattering $\rightarrow |\Delta \mathbf{v}_p|$ conserved
- Scattering angle = Rutherford scattering angle
- Limited by surface escape speed
- May need multiple scatterings (grav. assists) to achieve desired orbit
- Multiple assists require resonant orbits, in-out or out-in rendezvous orbits