Homework Exercise
Solar Convection and the Solar Dynamo

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PROBLEM 1: THERMAL WIND BALANCE

We begin with the equation that describes the conservation of momentum in a rotating fluid, given by

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \left(\mathbf{v} \cdot \boldsymbol{\nabla} \right) \mathbf{v} = -\boldsymbol{\nabla} P + \rho \boldsymbol{\nabla} \Psi - 2 \rho \boldsymbol{\Omega_0} \times \mathbf{v} \tag{1}$$

where ρ is the mass density, P is the gas pressure, $\mathbf{v}$ is the bulk velocity of the fluid in a reference frame rotating with an angular velocity of $\boldsymbol{\Omega_0}$, and Ψ is the effective gravitational potential. We have (justifiably) neglected magnetic fields and molecular viscosity here in order to illustrate an important dynamical balance for solar and planetary interiors and atmospheres. We will normally use spherical polar coordinates (r, θ, ϕ) but it is often instructive to convert the results into cylindrical coordinates (λ, ϕ, z) for interpretation, where $z = r \cos\theta$ and $\lambda = r \sin\theta$ represent the displacement parallel and perpendicular to the rotation axis, respectively (see Appendix). Thus, Ψ is given by

$$\Psi = \frac{GM}{r} + \frac{\lambda^2 \Omega_0^2}{2} \quad , \tag{2}$$

where G is Newton's gravitational constant, M is the mass internal to radius r, and Ω_0 is the amplitude of $\boldsymbol{\Omega_0}$ ($\boldsymbol{\Omega_0} = \Omega_0 \hat{\boldsymbol{z}}$).

(a) In rapidly rotating fluid systems[1] the Coriolis force on the right-hand-side (the term involving Ω_0) is often much larger than the fluid inertia relative to the rotating reference frame (the terms on the left-hand-side). Make this approximation now to derive the *thermal wind* equation:

$$2 \lambda \Omega_0 \frac{\partial \Omega}{\partial z} = \left\langle \frac{\boldsymbol{\nabla} P \times \boldsymbol{\nabla} \rho}{\rho^2} \right\rangle \cdot \hat{\boldsymbol{\phi}} \quad . \tag{3}$$

Here $\Omega = \Omega_0 + \langle v_\phi \rangle \lambda^{-1}$ is the total rotation rate, including both uniform and *differential* rotation, and angular brackets <> denote averages over longitude.

Proceed by taking the curl of equation (1), neglecting the inertial terms, and then average the zonal (ϕ) component over longitude. The expressions given in the Appendix should help.

(b) Consider the implications of equation (3). The term in brackets on the left-hand-side is known as the *baroclinic vector*. In many situations it can vanish. Examples include a polytropic equation of state given by $P \propto \rho^\gamma$ or an *isothermal* (constant T) or *isentropic* (constant S) stratification.

More generally, a *barotropic* equation of state or stratification is one in which the pressure is a function only of the density, $P = P(\rho)$. Sketch what the contours of Ω would look like if this

[1] Here "Rapidly rotating" means $R_o << 1$ where $R_o = U/(L\Omega_0)$ is the so-called *Rossby number*, and U and L are characteristic velocity and length scales of the motions.

were true in a spherical star, assuming that the equator rotates faster than the poles at the stellar surface. This is a solar/stellar manifestation of the *Taylor-Proudman theorem*.

(c) Now consider a hypothetical star with a cylindrical rotation profile, meaning that $\Omega = \Omega(\lambda)$, so angular velocity contours are parallel to the rotation axis. Assuming that equation (3) holds, this would mean that pressure and density surfaces must coincide *regardless of the equation of state*. Such a star is thus referred to as a *psuedo-barotrope*.

For the special case of a pseudo-barotrope, show that the Coriolis force arising from the differential rotation (which is essentially a centrifugal force) $2\boldsymbol{\Omega_0} \times (\langle v_\phi \rangle \hat{\boldsymbol{\phi}})$ can be written as the gradient of a potential and can thus be incorporated into equation (2); call the revised potential Ψ'. Furthermore, show that isobaric (constant P) surfaces coincide with geopotential (constant Ψ') surfaces.

(d) The specific entropy of an ideal gas is given by

$$S = C_P \ln\left(\frac{P^{1/\gamma}}{\rho}\right) \tag{4}$$

where $\gamma = C_V/C_P$ is the adiabatic index and C_P and C_v are the specific heats at constant pressure and density respectively.

In a stellar convection zone, the centrifugal and Coriolis forces are typically small relative to the gravitational acceleration so the radial component of $\boldsymbol{\nabla} P$ is much larger than the latitudinal component and is approximately given by hydrostatic balance:

$$\frac{\partial P}{\partial r} \approx -\rho g \tag{5}$$

where $g = GMr^{-2}$. Meanwhile, convection is good at mixing entropy so the radial and latitudinal components of $\boldsymbol{\nabla} S$ are comparable. Putting all these results together, show that the thermal wind equation (3) reduces to

$$\frac{\partial \Omega}{\partial z} = \frac{g}{2\Omega_0 r \lambda C_P}\frac{\partial \langle S \rangle}{\partial \theta} \tag{6}$$

List the assumptions we made to derive equation (6). Which of them are most essential?

(e) The rotation profile in the solar convection zone is approximately conical in the convection zone $\Omega \approx \Omega(r)$, decreasing monotonically from equator to pole. What does this imply about the sense of the entropy gradient, according to equation (6)?

PROBLEM 2: DYNAMO WAVES

For this problem we'll use Cartesian coordiates because the math is easier and they suffice to illustrate an important concept in dynamo theory.

Thus, consider the mean-field induction equation of magnetohydrodynamics in its simplest form

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times\left(\mathbf{v}\times\boldsymbol{B} + \alpha\boldsymbol{B}\right) \tag{7}$$

We have neglected turbulent and ohimc diffusion, magnetic pumping, and other terms in the mean-field expansion. Thus, $\mathbf{v}$ and $\boldsymbol{B}$ should be regarded as mean (large-scale) fields, with the α-effect arising from small-scale helical turbulence. We'll assume that this small-scale turbulence is homogeneous and quasi-isotropic[2] so α is a constant (which can be positive or negative).

Now consider a specified flow given by

$$\mathbf{v} = \Gamma z\hat{\boldsymbol{x}} \quad . \tag{8}$$

This describes a flow in the x or $-x$ direction (depending on the sign of Γ) whose amplitude increases with height z; in other words a *shear flow*. Now we will write the magnetic field as

$$\boldsymbol{B} = B_x\hat{\boldsymbol{x}} + \boldsymbol{\nabla}\times\left(A\hat{\boldsymbol{x}}\right) \tag{9}$$

and we will assume that B_x and A are independent of x; thus, they only depend on y, z, and time t. In fact, equation (9) is a completely general way to express a divergenceless vector field ($\boldsymbol{\nabla\cdot B} = 0$) that is independent of x.

(a) Substitute equations (8) and (9) into (7) to obtain evolution equations for the two unknowns, A and B_x:

$$\frac{\partial A}{\partial t} = \alpha B_x \tag{10}$$

$$\frac{\partial B_x}{\partial t} = -\Gamma\frac{\partial A}{\partial y} - \alpha\boldsymbol{\nabla}^2 A \tag{11}$$

Hint: Consider only the x component of (7) and the x component of the "uncurled" version of (7). Also, use the vector identities in the Appendix and note that $\nabla^2(A\hat{\boldsymbol{x}}) = (\nabla^2 A)\hat{\boldsymbol{x}}$.

(b) Now, for illustration, we are going to assume that the shear is strong enough that the Ω-effect dominates over the α-effect in equation (11). This amounts to setting $\alpha = 0$ in equation (11) but be sure to leave it non-zero in equation (10) – otherwise we wouldn't have a dynamo! Now derive the dynamo wave equation:

$$\frac{\partial^2 A}{\partial t^2} = -\alpha\Gamma\frac{\partial A}{\partial y} \quad . \tag{12}$$

(c) To demonstrate that this is indeed a wave equation, assume that A varies as

$$A(y,z,t) = \tilde{A}(z)e^{\imath(ky-\omega t)} \quad , \tag{13}$$

(with $k > 0$) and substitute it into (12) to obtain the dispersion relation

$$\omega = \pm\sqrt{\frac{|\alpha\Gamma|k}{2}}\;(s+\imath) \quad , \tag{14}$$

[2]Quasi-isotropic here means that the flow is not reflectionally symmetric but is otherwise the same in all directions.

where s is the sign of the product of α and Γ

$$s = \frac{\alpha\Gamma}{|\alpha\Gamma|} \quad . \tag{15}$$

Hint: $\sqrt{\imath} = (1+\imath)/\sqrt{2}$.

Consider the implications of equation (14). The frequency ω is complex, with both a real and imaginary part, implying that it has oscillatory solutions that either grow or decay exponentially with time. Chosing the positive sign in (14) selects the exponentially growing solution. The expansion in equation (13) implies that these growing oscillatory solutions are actually waves that propagate in the y direction (perpendicular to the flow $\mathbf{v}$ and the shear $\boldsymbol{\nabla} v_x$) with a phase speed given by the real part of ω:

$$\frac{\omega_r}{k} = s\sqrt{\frac{|\alpha\Gamma|}{2k}} \quad . \tag{16}$$

What would happen to the wave speed and direction if you were to double α? How would this effect the exponential growth rate? What if you keep the amplitude of α the same but reverse its sign? What if you were to keep α constant but change the sign and/or amplitude of Γ?

If we translate this to spherical coordinates, we can think of the x direction as longitude, the y direction as latitude, and the z direction. So, the shear flow $\mathbf{v}$ is analogous to differential rotation that varies with radius and the propagation is in the latitudinal direction. Given the solar internal rotation inferred from helioseimology, where might you expect such a wave to exist? *(Hint: look for where $\partial\Omega/\partial r$ is big).*

(d) Repeat the analysis in sections *(b)* and *(c)* for an α^2 dynamo. That is, set $\Gamma = 0$ in equation (11) and keep α non-zero. Is it still a dynamo wave?

Some stars exhibit magnetic cycles like the Sun and some do not. If we assume that the oscillatory behavior that characterizes a cycle is linked to a dynamo wave, then which do you think might be more likely to exhibit a magnetic cycle, an α-Ω dynamo or an α^2 dynamo? What does this (crude but instructive) model suggest about the potential role of differential rotation in the solar cycle?

(e) Only if you're feeling really ambitious! It is straightforward to repeat this entire analysis including ohmic and/or turbulent diffusion in equation (7). You'll find then that exponentially growing solutions (for the α-Ω case) can only occur if the dynamo number $\mathcal{D}$ is greater than one, where

$$\mathcal{D} = \frac{|\alpha\Gamma| k}{2\eta^2(k^2+k_z^2)^2} \quad . \tag{17}$$

Here η is the diffusivity (turbulent or ohmic) and k_z is the wavenumber in the z direction. To put it bluntly: too much diffusion can kill the dynamo.

APPENDIX

Here are some useful expressions, with $\mathbf{A}$ and $\mathbf{B}$ and f and g denoting arbitrary vector and scalar fields respectively. These should be sufficient for this problem set but feel free to consult a textbook for more.

Vector Identities

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla} f) = 0 \tag{18}$$

$$\boldsymbol{\nabla}(fg) = f\boldsymbol{\nabla} g + g\boldsymbol{\nabla} f \tag{19}$$

$$\boldsymbol{\nabla}\times(f\mathbf{A}) = f\boldsymbol{\nabla}\times\mathbf{A} - \mathbf{A}\times\boldsymbol{\nabla} f \tag{20}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2\mathbf{A} \tag{21}$$

$$\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + \mathbf{A}(\boldsymbol{\nabla}\cdot\mathbf{B}) - \mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{A}) \tag{22}$$

Vector Derivatives (spherical polar coordinates)

$$[(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B}]\cdot\hat{\boldsymbol{\phi}} = \mathbf{A}\cdot\boldsymbol{\nabla} B_\phi + \frac{A_\phi B_r}{r} + \frac{\cot\theta}{r}\, A_\phi B_\theta \tag{23}$$

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial f}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi} \tag{24}$$

$$\boldsymbol{\nabla}\times\mathbf{A} = \frac{\hat{\boldsymbol{r}}}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right] + \frac{\hat{\boldsymbol{\theta}}}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right] + \frac{\hat{\boldsymbol{\phi}}}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right] \tag{25}$$

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \tag{26}$$

Cylindrical coordinates

$$\lambda = r\sin\theta \qquad\qquad z = r\cos\theta \tag{27}$$

$$\hat{\boldsymbol{\lambda}} = \boldsymbol{\nabla}\lambda = \hat{\boldsymbol{r}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta \tag{28}$$

$$\hat{\boldsymbol{z}} = \boldsymbol{\nabla} z = \hat{\boldsymbol{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta \tag{29}$$