

Solutions to Merav Opher (2010) Problems

1. The normal of the shock is

$$\vec{n} = \frac{(\vec{B}_u \times \vec{B}_d) \times (\vec{B}_u - \vec{B}_d)}{|(\vec{B}_u \times \vec{B}_d) \times (\vec{B}_u - \vec{B}_d)|}$$

Since from the plot you can obtain all the three components of B_u and B_d , the normal can be easily found. The shock speed is:

$$V_s = \left(\frac{\rho_d \vec{u}_d - \rho_u \vec{u}_u}{\rho_d - \rho_u} \right) \cdot \vec{n}$$

With the shock Θ_{Bn} can be found. Alfven mach number, sonic mach number as well as the compression are easy and can be obtained directly using the graphs.

2. Considering a shock coordinate system where the parallel axis $\parallel$ is along the normal of the shock, $\hat{n} = \cos(\theta_1)\hat{R} + \sin(\theta_1)\cos(\phi_1)\hat{T} + \sin(\theta_1)\sin(\phi_1)\hat{N}$, where θ_1 and ϕ_1 are the angles between the normal $\mathbf{n}$ and the radial upstream velocity V_1 . θ_1 is defined as the angle measured between $\mathbf{n}$ and $\mathbf{R}$. ϕ_1 is the angle between the projection of $\mathbf{n}$ on the (T, N) plan and T (see Supplementary figure 1). The perpendicular axis are taken as $\hat{n}_{\perp 1} = \hat{T} \times \hat{n} / \|\hat{T} \times \hat{n}\|$, and $\hat{n}_{\perp 2} = \hat{n} \times \hat{n}_{\perp 1} / \|\hat{n} \times \hat{n}_{\perp 1}\|$.

The three-dimensional Rankine-Hugoniot conditions in this shock coordinate system (after a transformation to the deHoffman -Teller frame) are:

$$\begin{aligned} v_{2\parallel} &= \frac{v_{1\parallel}}{r} \\ v_{2\perp 1} &= v_{1\perp 1} \\ v_{2\perp 2} &= v_{1\perp 2} + \frac{B_{1\perp 2}}{B_{1\parallel}} \frac{(r-1)v_{1\parallel}v_{A1\parallel}^2}{(v_{1\parallel}^2 - rv_{A1\parallel}^2)} \end{aligned} \quad \text{Eqs. (1-3)}$$

where the index “1” and “2” refer respectively, to the upstream and downstream quantities; V_{A1} is the upstream Alfven speed. In the RTN frame Eqs. (1-3) become:

$$V_{2R} = \left(\frac{V_1 \cos(\theta_1) - V_s}{r} \right) \cos(\theta_1) + \frac{V_1 \sin^2(\theta_1) \sin^2(\phi_1)}{\alpha_1^2}$$

$$- \left(-\frac{V_1 \cos(\theta_1) \sin(\theta_1) \cos(\phi_1)}{\alpha_1} + \tan(\chi_1) \frac{(r-1)(V_1 \cos(\theta_1) - V_s)(V_{A1} \cos(\chi_1))^2}{((V_1 \cos(\theta_1) - V_s)^2 - r(V_{A1} \cos(\chi_1))^2)} \right) \frac{\cos(\theta_1) \sin(\theta_1) \cos(\phi_1)}{\alpha_1}$$

$$V_{2T} = \left(\frac{V_1 \cos(\theta_1) - V_s}{r} \right) \sin(\theta_1) \sin(\phi_1) +$$

$$+ \left(-\frac{V_1 \cos(\theta_1) \sin(\theta_1) \cos(\phi_1)}{\alpha_1} + \tan(\chi_1) \frac{(r-1)(V_1 \cos(\theta_1) - V_s)(V_{A1} \cos(\chi_1))^2}{((V_1 \cos(\theta_1) - V_s)^2 - r(V_{A1} \cos(\chi_1))^2)} \right) \alpha_1 \quad \text{Eqs.(4-6)}$$

$$V_{2N} = \left(\frac{V_1 \cos(\theta_1) - V_s}{r} \right) \sin(\theta_1) \sin(\phi_1) - \frac{V_1 \cos(\theta_1) \sin(\theta_1) \sin(\phi_1)}{\alpha_1^2}$$

$$\alpha = \frac{V_1 \cos(\theta_1) \sin(\theta_1) \cos(\phi_1)}{\sqrt{\sin^2(\theta_1) \sin^2(\phi_1) + \cos^2(\theta_1)}} \quad \text{is the angle between } \mathbf{B}_1 \text{ and the shock normal.}$$

$$\cos(\chi_1) = \sin(\theta_1) \cos(\phi_1) \text{ and } r \text{ is the compression ratio, } r = \rho_2 / \rho_1.$$

3. The Termination Shock at the time of the Voyager 2 crossing had, on average, for $\theta_1=15^\circ$, $\phi_1=165^\circ$.

4. The students should sketch the main MHD variables for a tangential discontinuity (heliopause) (so total pressure should be constant). They should also be able to sketch the jump at the termination shock. They should be able to compare Voyager 1 and 2 and qualitatively see that the magnetic pressure at Voyager 2 will be stronger therefore the heliopause will be closer to the sun than Voyager 1.

Shocks, Homework

5. a. The hydrodynamic jump conditions for plasma, $\gamma = 5/3$, are

$$M_2^2 = \frac{M_1^2 + 3}{5M_1^2 - 1} \rightarrow \frac{1}{5} \quad (1)$$

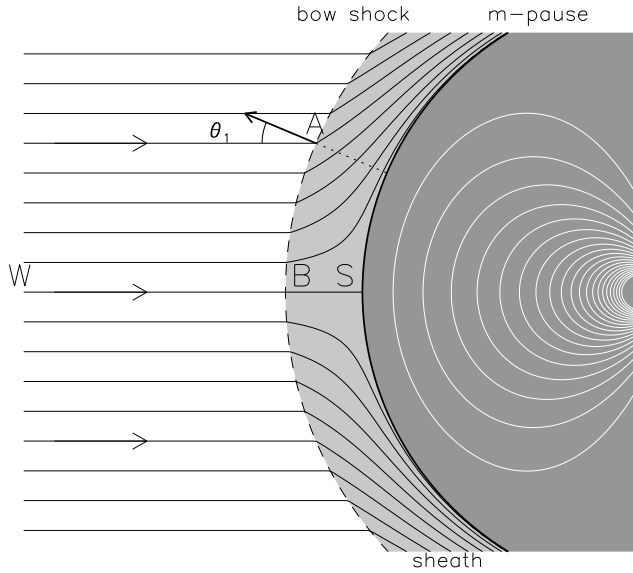
$$\frac{\rho_2}{\rho_1} = \frac{v_{n,1}}{v_{n,2}} = \frac{4}{1 + 3/M_1^2} \rightarrow 4 \quad , \quad (2)$$

where limits are for $M_1 \rightarrow \infty$. We can use these to write

$$\frac{p_2}{\rho_2} = \frac{v_2^2}{\gamma M_2^2} \rightarrow 3v_2^2 \rightarrow \frac{3}{16}v_1^2 = \frac{3}{16}v_{sw}^2 \quad . \quad (3)$$

This gives the temperature

$$T_B = \frac{m_p}{2k_B} \frac{p_B}{\rho_B} \simeq \frac{3m_p}{32k_B} v_{sw}^2 \quad . \quad (4)$$



A simplified of the day-side magnetosphere. Flow streamlines are shown as solid curves originating at the Sun, far to the left. The bow shock is a dashed arc and the magnetopause is a thick solid arc. The magnetosphere proper is dark grey, with some white magnetic field lines shown. The magnetosheath is the lighter gray region between the bow shock and the magnetopause.

- b. Using eq. (3) gives

$$\frac{1}{2}v_B^2 = \frac{1}{6} \frac{p_B}{\rho_B} \quad , \quad (5)$$

and thus

$$\frac{p_B}{\rho_B} + \frac{2}{5} \frac{1}{2}v_B^2 = \frac{16}{15} \frac{p_B}{\rho_B} = \frac{p_S}{\rho_S} \quad . \quad (6)$$

From this we find

$$T_S = \frac{16}{15}T_B = \frac{m_p}{10k_B} v_{sw}^2 = 1.2 \times 10^5 \text{ K} \left(\frac{v_{sw}}{100 \text{ km/s}} \right)^2 \quad . \quad (7)$$

Taking $v_{sw} = 800 \text{ km/s}$ gives $T_S = 7.7 \times 10^6 \text{ K}$.

c. It is clear from eq. (2) that

$$\rho_B = 4 \rho_{sw} . \quad (8)$$

Adiabatic compression leads to the ratio

$$\frac{\rho_S}{\rho_B} = \left(\frac{T_S}{T_B} \right)^{1/(\gamma-1)} = \left(\frac{T_S}{T_B} \right)^{3/2} = \left(\frac{16}{15} \right)^{3/2} = 1.102 . \quad (9)$$

From this we find

$$\rho_S \simeq \left(\frac{16}{15} \right)^{3/2} 4 \rho_{sw} = 4.407 \rho_{sw} . \quad (10)$$

d. The flow angle is defined

$$\tan \theta_j = \frac{v_{t,j}}{v_{n,j}} \quad (11)$$

where v_t , the flow tangent to the shock, is the same on both sides. This leads to

$$\tan \theta_2 = \frac{v_t}{v_2} \simeq 4 \frac{v_t}{v_1} = 4 \tan \theta_1 . \quad (12)$$

The deflection across the shock is

$$\Delta \theta = \theta_2 - \theta_1 \simeq 4 \theta_1 - \theta_1 = 3 \theta_1 , \quad (13)$$

after using the small angle approximation to replace $\tan \theta_j \simeq \theta_j$.

e. The post-shock velocity vector is

$$\mathbf{v}_2 = v_{n,2} \hat{\mathbf{n}} + v_{t,2} \hat{\mathbf{t}} = v_{n,2} [\hat{\mathbf{n}} + \tan \theta_2 \hat{\mathbf{t}}] = v_{n,2} [\hat{\mathbf{n}} + 4 \tan \theta_1 \hat{\mathbf{t}}] \quad (14)$$

where $\hat{\mathbf{n}}$ is the shock normal and $\hat{\mathbf{t}}$ a tangent vector perpendicular to it. The square magnitude is therefore

$$|\mathbf{v}_2|^2 = v_{n,2}^2 [1 + 16 \tan^2 \theta_1] . \quad (15)$$

Compared to the local sound speed this is

$$\frac{|\mathbf{v}_2|^2}{c_{s,2}^2} = \frac{v_{n,2}^2}{c_{s,2}^2} [1 + 16 \tan^2 \theta_1] = M_2^2 [1 + 16 \tan^2 \theta_1] = \frac{1}{5} [1 + 16 \tan^2 \theta_1] ,$$

using the hypersonic limit from eq. (1). Setting this to exceed one gives the requirement

$$\tan \theta_1 > \frac{1}{2} , \quad \theta_1 > \tan^{-1}(1/2) = 26.5^\circ . \quad (16)$$

f. The radial and poloidal velocities are

$$v_r = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} = 2 \frac{A}{R_{mp}^2} \left(\frac{r^2}{R_{mp}^2} - \frac{R_{mp}^3}{r^3} \right) \cos \theta , \quad (17)$$

$$v_\theta = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} = -\frac{A}{R_{mp}^2} \left(4 \frac{r^2}{R_{mp}^2} + \frac{R_{mp}^3}{r^3} \right) \sin \theta , \quad (18)$$

so evidently A is negative. The angle of the downstream flow relative to the shock normal, $\hat{\mathbf{n}} = \hat{\mathbf{r}}$, at $r = R_{bs}$, just inside the bow shock, is

$$\tan \theta_2 = -\frac{v_\theta}{v_r} = \frac{4(R_{bs}/R_{mp})^3 + (R_{mp}/R_{bs})^2}{2(R_{bs}/R_{mp})^3 - 2(R_{mp}/R_{bs})^2} \tan \theta . \quad (19)$$

According to part d. this must be $\tan \theta_2 = 4 \tan \theta_1$, and $\theta_1 = \theta$, to polar angle, since the flow is horizontal outside the shock. This leads to the relation

$$\frac{4(R_{bs}/R_{mp})^5 + 1}{2(R_{bs}/R_{mp})^5 - 2} = 4 , \quad (20)$$

and therefore

$$R_{bs} = \left(\frac{9}{4}\right)^{1/5} R_{mp} = 1.18 R_{mp} . \quad (21)$$

This result is often cast in terms of the “stand-off” distance between the bow shock and magnetopause:

$$\Delta = R_{bs} - R_{mp} = 0.18 R_{mp} , \quad M_{sw} \rightarrow \infty . \quad (22)$$

The simple model proposed above, a super-sonic flow encountering a spherical obstacle of radius R , is one for which there has been much study. Laboratory experiments with different Mach numbers and different fluids (i.e. differing γ) have led to empirical relations of the form

$$\frac{\Delta}{R} \simeq \alpha \frac{\rho_1}{\rho_2} , \quad (23)$$

with values $\alpha \simeq 0.78$ (Seiff, *NASA Tech. Pub.* 24, [1962]). Numerical solutions of fully compressible hydrodynamics yield $\alpha \simeq 1.1$ (Spreiter, Summers & Alkse, *Planet. Space Sci.* 14 223 [1966]), which has been subsequently used to predict the stand-off distance of the actual bow shock, at least for cases with high solar wind Mach number (see Farris & Russell, *JGR* 99, 17681 [1994]). The simplified calculation we have performed here has found a lower value, $\alpha = 0.18 \times 4 = 0.704$, due mostly to the departure of actual post-shock flow from the incompressible solution we assumed. It is not, however, due to our assumption of a spherical magnetopause. As shown in part e., the sheath flow becomes supersonic beyond an angle of $\theta = 26^\circ$. Inside this small region there is little difference between a sphere and the actual magnetopause. Differences outside that region can affect neither the subsonic solution nor the stand-off distance.

6. a. The velocity difference is equal to v_f , and at high Mach number this difference is

$$v_f = v_1 - v_2 = \left(1 - \frac{v_2}{v_1}\right) v_1 \simeq \frac{3}{4} v_1 . \quad (24)$$

Since the pre-shock material is at rest relative to the Sun, the shock moves downward at speed v_1

$$v_s = v_1 = \frac{4}{3} v_f . \quad (25)$$

- b. The post-shock velocity, in the shock frame, is

$$v_2 = \frac{1}{4} v_1 = \frac{1}{3} v_f . \quad (26)$$

Using the fact that $v_2/c_{s,2} = M_2 = 1/\sqrt{5}$ we find that

$$\frac{1}{\sqrt{5}} = \frac{1}{3} \frac{v_f}{c_{s,f}} \implies c_{s,f} = \frac{\sqrt{5}}{3} v_f \quad (27)$$

The downward flows is therefore related to the post-shock temperature

$$T_f = \frac{3}{5} \frac{\bar{m}}{k_B} c_{s,2}^2 = \frac{1}{3} \frac{\bar{m}}{k_B} v_f^2 = 2 \times 10^7 \text{ K} \left(\frac{v_f}{100 \text{ km/s}} \right)^2 . \quad (28)$$

- c. After reflection there is a new shock propagating upward into the flare plasma. The velocity difference is the same, $v_1 - v_2 = v_f$, but the pre-shock plasma of this shock is the post-shock plasma from the incident shock: the flaring plasma. This is not, however, a high Mach-number shock so

$$\begin{aligned} v_f &= v_1 - v_2 = \left(1 - \frac{v_2}{v_1}\right) v_1 = \left(1 - \frac{2M_1^2 + 6}{8M_1^2}\right) v_1 \\ &= c_{s,f} M_1 \left(\frac{3}{4} - \frac{3}{4} M_1^{-2}\right) = \frac{3}{4} c_{s,f} (M_1 - M_1^{-1}) . \end{aligned} \quad (29)$$

The results of the previous section therefore lead to the condition

$$\frac{3}{4} (M_1 - M_1^{-1}) = \frac{v_f}{c_{s,f}} = \frac{3}{\sqrt{5}} . \quad (30)$$

This is a quadratic equations whose positive solution is $M_1 = \sqrt{5}$. This is the Mach number of the reflected shock.

- d. The reflected shock enhances the pressure by

$$\frac{p_2}{p_1} = \frac{5M_1^2 - 1}{4} = 6 , \quad (31)$$

above the pressure following the first, hypersonic, shock. This pressure is

$$\tilde{p}_2 = \frac{3}{4} \tilde{\rho}_1 \tilde{v}_1^2 = \frac{4}{3} \rho_0 v_f^2 , \quad (32)$$

where ρ_0 is the pre-flare coronal density. The the final chromospheric pressure is

$$p_{\text{chr.}} = 6 \times \frac{4}{3} \rho_c v_f^2 = 8 \rho_0 v_f^2 . \quad (33)$$

e. The transmitted shock raises the pressure by

$$\frac{p_2}{p_1} = \frac{8 \rho_0 v_f^2}{p_0} \simeq \frac{5}{4} M_1^2 , \quad (34)$$

where p_0 and ρ_0 are pre-flare coronal values; the pre-flare chromosphere is at the same pressure as the pre-flare corona since they are separated by a *contact discontinuity*. Using the fact that $p_0/\rho_0 = (3/5)c_{s,0}^2$ this becomes

$$\frac{40v_f^2}{3c_{s,0}^2} = \frac{5}{4} M_1^2 , \quad (35)$$

from which we see that $M_1 = \sqrt{32/3}(v_f/c_{s,0})$. It does not depend on r .

f. Since the pre-flare chromosphere is colder than the pre-flare corona by $1/r$, its sound speed is $c_{s,0}/\sqrt{r}$. This is the sound speed by which the pre-shock speed, v_1 , is divided to form M_1 . The velocity difference is then

$$\frac{v_1 - v_2}{c_{s,0}/\sqrt{r}} = \frac{3}{4} \frac{v_1}{c_{s,0}/\sqrt{r}} = \frac{3}{4} M_1 = \sqrt{6} \frac{v_f}{c_{s,0}} . \quad (36)$$

This shows that the velocity jump across the transmitted shock is

$$v_1 - v_2 = \sqrt{\frac{6}{r}} v_f . \quad (37)$$

Since the pre-shock material is at rest this is the downward velocity of the transmitted shock. In order to neglect this velocity in the reflected shock we need $\sqrt{6/r} \ll 1$.

	17:25UT (first)	17:49UT (second)
n [cm ⁻³]	11.62	4.25
v_R [km/s]	349.5	295.9
v_T [km/s]	64.80	79.14
v_N [km/s]	37.98	59.75
B_R [nT]	0.027	1.000
B_T [nT]	-1.759	-0.908
B_N [nT]	-2.931	-1.474

7. a. A shock always *compresses* the plasma so the post-shock region will have the higher density — the **second observation is pre-shock**. This is the second region to pass the spacecraft, and therefore lies at smaller radius than the post-shock region. The shock normal $\hat{\mathbf{n}}$ must point *toward* the pre-shock region so $\hat{\mathbf{n}} \cdot \hat{\mathbf{r}} < 0$. This is therefore a *reverse shock*.
- b. Taking the dot product with $\hat{\mathbf{n}}$ shows

$$\hat{\mathbf{n}} \cdot [\mathbf{B}] = [\hat{\mathbf{n}} \cdot \mathbf{B}] = [B_n] = 0, \quad (38)$$

from the jump condition following from $\nabla \cdot \mathbf{B} = 0$. This shows that $[\mathbf{B}]$ is orthogonal to $\hat{\mathbf{n}}$.

- c. The jump in velocity is

$$[\mathbf{v}] = \hat{\mathbf{n}}[v_n] + [\mathbf{v}_\perp] = \hat{\mathbf{n}}[v_n] + \frac{B_n}{4\pi\rho v_n}[\mathbf{B}_\perp] = \hat{\mathbf{n}}[v_n] + \frac{B_n}{4\pi\rho v_n}[\mathbf{B}]. \quad (39)$$

In the special case that $[v_n] = 0$, the two jumps $[\mathbf{v}]$ and $[\mathbf{B}]$, lie along the same line (they are parallel or anti-parallel). This condition is equivalent to $[\rho] = 0$, which is a rotational discontinuity. This does not apply here since $n_{p1} \neq n_{p2}$. Thus the velocity jump is a sum of two perpendicular components and lies in the plane spanned by them: $[\mathbf{B}]$ and $\hat{\mathbf{n}}$.

- d. First we compute the differences directly

$$\Delta\mathbf{B} = (-0.97, -0.85, -1.46) \mu\text{G}, \quad \Delta\mathbf{v} = (53.57, -14.34, -21.77) \text{ km/s}$$

whose magnitudes are

$$|\Delta\mathbf{B}| = 1.95 \mu\text{G}, \quad |\Delta\mathbf{v}| = 59.58 \text{ km/s}$$

We can extract from $\Delta\mathbf{v}$ that component perpendicular to $\Delta\mathbf{B}$

$$\begin{aligned} \mathbf{w} &= \Delta\mathbf{v} - \frac{\Delta\mathbf{v} \cdot \Delta\mathbf{B}}{|\Delta\mathbf{B}|^2} \Delta\mathbf{B} \\ &= \Delta\mathbf{v} - \frac{-8.21}{3.79} \Delta\mathbf{B} = (51.47, -16.18, -24.92) \text{ km/s} \end{aligned} \quad (40)$$

The normal vector $\hat{\mathbf{n}}$ must have unit magnitude and a negative radial component

$$\hat{\mathbf{n}} = -\frac{\mathbf{w}}{|\mathbf{w}|} = (-0.866, 0.272, 0.419)$$

- e. The dot product of the normal vector with the magnetic fields gives the angles

$$\cos \theta_1 = \frac{\mathbf{B}_1 \cdot \hat{\mathbf{n}}}{|\mathbf{B}_1|} = \frac{-1.73}{2.00} = -0.866 \quad , \quad \theta_1 = 150.000^\circ$$

$$\cos \theta_2 = \frac{\mathbf{B}_2 \cdot \hat{\mathbf{n}}}{|\mathbf{B}_2|} = \frac{-1.73}{3.42} = -0.507 \quad , \quad \theta_2 = 120.438^\circ$$

- f. From the expressions we find

$$M_{A2}^2 = \frac{n_1}{n_2} M_{A1}^2$$

and

$$(M_{A1}^2 - 1) \tan \theta_1 = (M_{A2}^2 - 1) \tan \theta_2 = \left(\frac{n_1}{n_2} M_{A1}^2 - 1 \right) \tan \theta_2$$

Solving this gives

$$M_{A1} = \sqrt{\frac{\tan \theta_1 - \tan \theta_2}{\tan \theta_1 - (n_1/n_2) \tan \theta_2}}$$

$$M_{A1} = 5.000 \quad , \quad M_{A2} = 3.024$$

- g. We have several pieces of evidence that this is a **fast shock**. We could have noted early on that $|\mathbf{B}_2| > |\mathbf{B}_1|$. Later it was evident that $\mathbf{B}_2$ had deflected *away* from the shock normal (although it appears that $\theta_2 < \theta_1$ because the field is directed backwards). Finally, we see that the Mach numbers are ordered

$$1 < M_{A2} < M_{A1} \quad , \quad (41)$$

which is a prerequisite for the *fast shock*.

- h. The Alfvén Mach number refers to the normal component of the upstream velocity $v_{1,n}$ in the reference frame of the shock. If we denote the shock speed v_s the inflow speed is therefore

$$v'_{n,1} = v_{n,1} - v_s = -M_{A1} v_{A1,n} \quad , \quad (42)$$

since the inflow is always in the direction opposite $\hat{\mathbf{n}}$. The shock speed is therefore

$$v_s = v_{n,1} + M_{A1} v_{A1,n} \quad , \quad (43)$$

where $v_{1,n}$ is the component of the velocity, in the spacecraft frame,

$$v_{n,1} = \hat{\mathbf{n}} \cdot \mathbf{v}_1 = -2.10 \times 10^7 \text{ cm/s} = -210 \text{ km/s} \quad .$$

The normal component of the Alfvén velocity is

$$v_{An,1} = \frac{|\mathbf{B}_1 \cdot \hat{\mathbf{n}}|}{\sqrt{4\pi n_1 m_p}} = \frac{1.73 \times 10^{-5} G}{9.44 \times 10^{-12}} = 1.83 \times 10^6 \text{ cm/s} = 18.3 \text{ km/s} \quad .$$

This gives the shock velocity in the reference frame of the spacecraft

$$v_s = (-210 + 5.00 \times 18.3) \text{ km/s} = -118 \text{ km/s} \quad , \quad (44)$$

The shock propagates in the direction of the shock normal giving

$$\mathbf{v}_s = \hat{\mathbf{n}} v_s = (103, -32.2, -49.7) \text{ km/s} \quad . \quad (45)$$

The shock is therefore moving *outward* from the Sun, even though it is a *reverse shock*.